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OPTIMAL CONTROL OF FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

George Alan Kent

Optimal Control of Functional Differential Equations of Neutral Type

by

George Alan Kent

B.S., United States Naval Academy, 1965

Thesis

submitted in partial fulfillment of the requirements for the
Degree of Doctor of Philosophy in the Division of
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Vita

George Alan Kent was born in Ames, Iowa on June 27, 1943. He attended schools primarily in Ithaca, New York, and in June 1961 graduated from Ithaca High School. In June 1965 he graduated from the United States Naval Academy with a major in Mathematics. After duty in submarines, he began graduate work in the Division of Applied Mathematics at Brown University in September 1967. He became a full member of Sigma Xi in May 1970.

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1. Introduction

In the following chapters we prove various results for control of systems governed by functional differential equations of neutral type. These equations are generalizations of the equation

$$\dot{x}(t) + A\dot{x}(t-h) = Bx(t) + Cx(t-h) + Du(t),$$

which has been studied extensively (for example, see Bellman and Cooke [4]).

Chapter 2 contains results in the theory of neutral equations. The form of the left-hand side of equation (2.1) given by (2.3) was first used in Hale and Meyer [14]. The type of hereditary dependence employed here occurs especially in integral equations; although this dependence puts a different topology on the domain of the functions than that used in the original proofs of the main theorems, the change does not alter the proofs significantly. Needless to say, the work of Professor Hale and his associates has greatly influenced many of the ideas and proofs in this thesis.

In chapter 3 we develop the concepts which will be used to prove necessary conditions. The definitions of $C(I,X)_{\Psi,K}$ and of a quasi-convex family are extensions of the definitions of $AC(I,X)_K$ and of an absolutely quasi-convex family given in Banks [1]. The general approach of the proof of theorem 3.1 follows a proof given by Neustadt [21; theorem 3.1] in the case of ordinary differential equations; Neustadt gives credit to Gamkrelidze [10; theorem 2.1] for many of the ideas of the proof.

Chapter 4 contains three control problems and the corresponding necessary conditions. In addition, results for the third problem from a slightly different approach are stated without proof. The proofs use the properties of quasi-convex families in a manner similar to that of the proofs of Neustadt in [21] and [22], rather than the techniques of the Hamiltonian func-

tion and variations as used by Pontryagin et al. in [23]. Kamenskii and Khvilon [19] use the latter method to obtain necessary conditions for the equation

$$\dot{x}(t) = f(x(t), x(h(t)), \dot{x}(h(t)), u(t)).$$

Although they allow nonlinearities in $\dot{x}(h(t))$, the rest of their assumptions are much more restrictive than those made below. Theorem 4.3 describes necessary conditions for driving a solution of the system equations to a terminal function rather than a terminal point; the author has not found any other such conditions in the literature. Although theorem 4.3 does not give conditions under which the multiplier ψ is non-zero on a set of positive measure, chapter 6 contains several examples with non-trivial ψ.

The sufficiency conditions of chapter 5 are proved by a method first used by Rozonoer [24] for ordinary differential equations and cost functions which are linear in x. He pointed out [24; pp. 1412, 1420] the necessity of assuming $\alpha^0 \neq 0$ in the case of a restrained terminal point. Halanay [11] extended Rozonoer's result to retarded equations in the case of a free endpoint. Lee [20] obtained a more general sufficiency theorem for ordinary differential equations. In parts (B) and (C) of that theorem he states explicitly that $\alpha^0 = -1$; although he does not state $\alpha^0 \neq 0$ in part (A), his proof uses a lemma which requires $\alpha^0 < 0$. All of the above results except part (C) of Lee's theorem are included in theorem 5.1. Theorems 5.3 and 5.4 on the existence of optimal controls are generalizations, from ordinary differential equations to neutral equations, of theorems 4.1 and 4.2 of Jacobs [18]. These are included to show a class of problems for which an optimal control exists; no attempt has been made to prove the most general existence theorem.

The first part of chapter 6 shows the relation between certain types of hyperbolic partial differential equations with boundary controls and neutral equations with control. Other authors have shown that some hyperbolic partial differential equations can be transformed into neutral equations (see [5] and [7]). Theorem 6.1 is a specialization of theorem 4.3 to equations with one fixed lag and constant coefficients, a fixed initial function, and a fixed final time. It can be understood independently of most of the rest of the thesis, requiring only some knowledge of such equations, lemma 3.3 (conditions for a family of functions to be quasiconvex), and remark 4.6 (on non-triviality of the multipliers). The last part of the chapter contains examples for which problem 4.3 is normal and theorems 4.3 (or 6.1) and 5.2 prove certain controls are optimal.

2. Neutral Equations

We will work on an interval $[\alpha_0, a)$, where $-\infty < \alpha_0 < t_0 < a \leq \infty$. A function $h(x(\cdot),t)$ may depend on any or all values $x(s)$, $\alpha_0 \leq s \leq t$. On $C([\alpha_0,t],R^n)$ we use the norm $\|\psi\|_t = \sup_{\alpha_0 \leq s \leq t} |\psi(s)|$ (the subscript t may be ommitted). Throughout the discussion the following convention holds: If $\psi \in C([\alpha_0,\tau],R^n)$, $t_0 \leq \tau < t < a$, we also consider $\psi \in C([\alpha_0,t],R^n)$ by setting $\psi(s) = \psi(\tau)$ for $s \in [\tau,t]$. x_σ denotes the restriction of x to $[\alpha_0,\sigma]$. We note that this convention means for $t_0 \leq \sigma < t < a$, x and x_σ are both elements of $C([\alpha_0,t],R^n)$, but unless x is constant on $[\sigma,t]$ they are different functions.

We consider the equation

$$\frac{d}{dt}[D(x(\cdot),t)] = f(x(\cdot),t) \tag{2.1}$$

together with the initial condition

$$x(t) = \varphi(t) \quad \text{on} \quad [\alpha_0,t_0]. \tag{2.2}$$

We shall consider y to be a (local) solution of (2.1), (2.2) if there is a τ, $t_0 < \tau < a$, such that y satisfies (2.2) and satisfies (2.1) a.e. on $[t_0,\tau]$. Unless otherwise stated we also require a solution y to be in $C([\alpha_0,\tau],R^n)$.

The following standing assumptions are made:

$$D(x(\cdot),t) = x(t) - g(x(\cdot),t) = x(t) - \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]x(\theta) \tag{2.3}$$

for $t \in [t_0,a)$, where

(2.4) $\mu(\sigma,\theta) = 0$ for $\theta \geq \sigma$, $\mu: R^2 \to R^{n^2}$; μ Borel-measurable; μ continuous from the right in its first argument and continuous from the left in its second argument; μ of bounded variation on every finite interval in its second argument, and such that $g(\psi,s)$ is continuous in $s \in [t_0,a)$ for each $\psi \in C([\alpha_0,a),R^n)$, (hence, continuous in (ψ,s) jointly).

(2.5) there exists a continuous non-decreasing function δ such that, for each $t \in [t_0,a)$, for each $\varepsilon \in [0,t-\alpha_0]$, $\underset{[t-\varepsilon,t]}{\mathrm{Var}}\ \mu(t,\cdot) \leq \delta(\varepsilon)$, and $\delta(0) = 0$.

(2.6) $f(\psi,t)$ is measurable in t for each fixed $\psi \in C([\alpha_0,a),R^n)$, is continuous in ψ for each fixed t, and for each compact $X \subset R^n$, there exists $k(\cdot) \in L^1_{loc}([t_0,a),R)$ such that $|f(\psi,s)| \leq k(s)$ for all $s \in [t_0,a)$, $\psi \in C([\alpha_0,a),X)$.

Remark 2.1. One condition which is sufficient to guarantee that $g(\psi,s)$ is continuous in s over $[t_0,a)$ for fixed $\psi \in C([\alpha_0,a),R^n)$ is that $\mu(s,\theta) = J(s,\theta) + \nu(s,\theta)$, where

a) $J(s,\theta) = \sum_{\ell=1}^{p} a_\ell(s)X[h_\ell(s)](\theta)$; $a_\ell: R \to R^{n^2}$ and $h_\ell: R \to R$ are continuous, $\alpha_0 \leq h_\ell(s) < s$, $s \in [t_0,a)$, $\ell = 1,\ldots,p$; and $X: R \to BV(R,R^{n^2})$ is given by

$$X[h](t) = \begin{cases} -E, & t \leq h \\ 0, & t > h \end{cases}$$

(E is the n × n identity matrix).

b) for each $s \in [t_0,a)$, $\nu(s,\cdot) \in C([\alpha_0,a),R^{n^2})$, and the map $s \to \nu(s,\cdot): R \to BV([\alpha_0,a),R^{n^2})$ is continuous (note that this last condition, combined with $\nu(s,s) = 0$, implies $\nu(\cdot,\theta)$ continuous).

Proof: Let $\varepsilon > 0$, $\psi \in C([\alpha_0,a),R^n)$ be fixed. Let $s_0 \in [t_0,a)$, consider $s \in [t_0,a)$.

From a), there exists $\rho_1 > 0$ such that $|s-s_0| < \rho_1$ implies $|a_\ell(s_0)-a_\ell(s)| < \frac{\varepsilon}{[(2p+1)\|\psi\|]}$, $\ell = 1,\ldots,p$.

From continuity of the h_ℓ and ψ (hence equicontinuity on $[t_0,s_0+\rho_1]$), there exists ρ_2, $0 < \rho_2 \leq \rho_1$, such that $|s-s_0| < \rho_2$ implies

$$|\psi(h_\ell(s_0))-\psi(h_\ell(s))| < \frac{\varepsilon}{[(2p+1) \max\limits_{\ell=1,\ldots,p} |a_\ell(s_0)|]}, \quad \ell = 1,\ldots,p.$$

From b), there exists ρ_3, $0 < \rho_3 \leq \rho_2$, such that $|s-s_0| < \rho_3$ implies

$$\operatorname*{var}_{[\alpha_0,a)} [\nu(s_0,\cdot)-\nu(s,\cdot)] < \frac{\varepsilon}{[(2p+1)\|\psi\|]}.$$

Thus, for $|s-s_0| < \rho_3 \leq \rho_2 \leq \rho_1$,

$$|g(\psi,s_0)-g(\psi,s)| = \left|\int_{\alpha_0}^{a} d_\theta[\mu(s_0,\theta)]\psi(\theta) - \int_{\alpha_0}^{a} d_\theta[\mu(s,\theta)]\psi(\theta)\right|$$

$$= \left|\int_{\alpha_0}^{a} d_\theta[\nu(s_0,\theta)-\nu(s,\theta)]\psi(\theta)\right.$$

$$\left.+ \Sigma_{\ell=1}^{p} a_\ell(s_0)\psi(h_\ell(s_0)) - \Sigma_{\ell=1}^{p} a_\ell(s)\psi(h_\ell(s))\right|$$

$$\leq \|\psi\| \operatorname*{var}_{[\alpha_0,a)} [\nu(s_0,\cdot)-\nu(s,\cdot)] + \Sigma_{\ell=1}^{p}|a_\ell(s_0)||\psi(h_\ell(s_0))-\psi(h_\ell(s))|$$

$$+ \Sigma_{\ell=1}^{p}|a_\ell(s_0)-a_\ell(s)||\psi(h_\ell(s))|$$

$$< \|\psi\|\frac{\varepsilon}{[(2p+1)\|\psi\|]} + \Sigma_{\ell=1}^{p}|a_\ell(s_0)|\frac{\varepsilon}{[(2p+1) \max\limits_{\ell=1,\ldots,p} |a_\ell(s_0)|]}$$

$$+ \Sigma_{\ell=1}^{p} \frac{\varepsilon}{[(2p+1)\|\psi\|]}\|\psi\|$$

$$\leq \varepsilon.$$

Theorem 2.1. Given (2.3)-(2.6), open $\Omega \subset C([\alpha_0,a),R^n)$, $\varphi \in C([\alpha_0,t_0],R^n) \cap \Omega$; there exists a solution of (2.1), (2.2) over $[\alpha_0,t_0+\gamma]$ for some $\gamma \in (0,a-t_0)$.

Proof: The proof is the same as that of Hale and Cruz [13] theorem 4.1, in light of the remarks made in section 7 of that reference and the arguments given in Hale [12], pp. 15, 16.

Theorem 2.2. Given (2.3)-(2.6), Ω open in $[\alpha_0,a) \times C([\alpha_0,a),R^n)$, x a noncontinuable solution of (2.1), (2.2) on $[\alpha_0,b)$, $t_0 < b < a$, $(t_0,\varphi) \in \Omega$. Then, if $W = \text{closure}\{(t,x_t): t \in [t_0,b)\}$ is compact, there exists a sequence $t_k \to b^-$ as $k \to \infty$ such that $(t_k,x_{t_k}) \to \partial\Omega$ as $k \to \infty$.

Proof: The proof is the same as that of Hale and Cruz [13], theorem 5.1.

Theorem 2.3. Given (2.3)-(2.7), Ω open in $[\alpha_0,a) \times C([\alpha_0,a),R^n)$, x a noncontinuable solution of (2.1), (2.2) on $[\alpha_0,b)$, $t_0 < b < a$, $(t_0,\varphi) \in \Omega$. Then, for any compact set $U \subset \Omega$, there exists $t_u \in (t_0,b)$ such that $(t,x_t) \notin U$

for $t_u \leq t < b$.

Proof: One proof of this has been given by Hale, see [12], theorem 5.2.

We now turn to linear equations, where

$$(2.7) \qquad f(x(\cdot),t) = \int_{\alpha_0}^{t} d_\theta[\eta(t,\theta)]x(\theta), \quad t \in [t_0,a)$$

under the conditions

(2.8) $\eta(\sigma,\theta) = 0$ for $\theta \geq \sigma$, η: $R^2 \to R^{n^2}$ is measurable, of bounded variation on every finite interval in its second argument, $\eta(\sigma,\cdot)$ left-continuous except at σ, and there exists $m \in L^1_{loc}([t_0,a),R)$ such that $\operatorname*{var}_{[\alpha_0,\sigma]} \eta(\sigma,\cdot) \leq m(\sigma)$.

Lemma 2.1. Given (2.4), (2.5), (2.8), the equations

$$(2.9) \qquad Y(s,t) = E + \int_s^{t^+} d_\alpha[Y(\alpha,t)]\mu(\alpha,s) - \int_s^t Y(\alpha,t)\eta(\alpha,s)d\alpha, \quad s < t$$

$$(2.10) \qquad Y(t,t) = E$$

$$(2.11) \qquad Y(s,t) = 0, \quad s > t$$

uniquely define a function Y: $R^2 \to R^{n^2}$ such that $Y(s,t)$ is left-continuous in s. In addition, for every $M > 0$, there exists a B_Y (depending on M) such that for $|t| \leq M$, $|t_0| \leq M$, $\operatorname*{var}_{[t_0,\infty)} Y(\cdot,t) \leq B_Y$ and $|Y(t_0,t)| \leq B_Y$.

Proof: The proof of the existence of $Y(s,t)$ is the same as that of Henry [15], lemma 1. The proof of the bound, due to Banks, is contained in Banks and Kent [13].

Theorem 2.4. Let $x = x(\varphi,h)$ be the solution of

$$(2.12)\qquad \begin{aligned} x(t) &= D(\varphi,t_0) + g(x(\cdot),t) + \int_{t_0}^{t} f(x(\cdot),\alpha)d\alpha + \int_{t_0}^{t} h(\alpha)d\alpha, \quad t \geq t_0 \\ x_{t_0} &= \varphi \end{aligned}$$

where D,f,g satisfy (2.3)-(2.5), (2.7), (2.8), $h \in L^1_{loc}([t_0,\infty),R^n)$. Then, for $t \geq t_0$

$$(2.13)\qquad \begin{aligned} x(t) = Y(t_0,t)D(\varphi,t_0) &+ \int_{\alpha_0}^{t_0^-} d_\beta\{-\int_{t_0}^{t^+} d_\alpha[Y(\alpha,t)]\mu(\alpha,\beta) \\ &+ \int_{t_0}^{t} Y(\alpha,t)\eta(\alpha,\beta)d\alpha\}\varphi(\beta) + \int_{t_0}^{t} Y(\alpha,t)h(\alpha)d\alpha \end{aligned}$$

where $Y(s,t)$ is given by (2.9)-(2.11).

Proof: The proof is similar to that of Henry [15], theorem 2.

Remark 2.2. The proof does not use the continuity of x and φ, so they may be taken to be discontinuous functions in $BV([\alpha_0,a),R^n)$ as well as continuous functions; the theorem will still hold. (See Lemma 2.7).

Lemma 2.2. Given (2.4), (2.5), (2.8)-(2.11), $Y(s,t)$ satisfies

$$(2.14)\qquad Y(s,t) = E + \int_{s}^{t} d_\theta[\mu(t,\theta)]Y(s,\theta) + \int_{s}^{t}\int_{s}^{\lambda} d_\theta[\eta(\lambda,\theta)]Y(s,\theta)d\lambda$$

for $t_0 \leq s \leq t$.

Proof: Using the methods of Hale and Meyer [14], theorem 2, from theorem 2.4 above we obtain that, if $W(t,s) = -\int_{s}^{t} Y(\theta,t)d\theta$, then $W(t,s) = 0$ for $s > t$, and for $s \leq t$

$$W(t,s) = \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]W(\theta,s) + \int_{t_0}^{t}\int_{\alpha_0}^{\lambda} d_\theta[\eta(\lambda,\theta)]W(\theta,s)d\lambda - (t-s)E.$$

By the bound in lemma 2.1, we may differentiate with respect to s under the integrals, obtaining (2.14) after noting that, for $s > t$, $Y(s,t) = 0$.

Lemma 2.3. Suppose for some integer M we are given two (scalar) functions on $[-M,M] \times [-M,M]$, $X(s,t)$ and $m(t,s)$. Let X be Borel-measurable in (s,t), of bounded variation and left-continuous in s for each fixed t, $\underset{[-M,M]}{\mathrm{var}}\ X(\cdot,t) \leq K$, all $t \in [-M,M]$, $X(s,t) = 0$, $s > t$. Let m satisfy (2.4), (2.5), and be of bounded variation in t for each fixed $s \in [-M,M]$. Then $\int_{-M}^{M+} d_\alpha[X(\alpha,t)]m(\alpha,s)$ is Borel-measurable in (t,s).

Proof: There exist Borel-measurable $m^+(t,s)$, $m^-(t,s) \geq 0$, non-decreasing in t for each fixed s, $m^+(t,s) = m^-(t,s) = 0$, $s \geq t$, $\underset{[-M,M]}{\mathrm{var}}\ m^{\pm}(t,\cdot) \leq \underset{[-M,M]}{\mathrm{var}}\ m(t,\cdot) \leq \delta(2M)$, and $m(t,s) = m^+(t,s) - m^-(t,s)$. Hence

$$\text{a)}\quad \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m(\alpha,s) = \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m^+(\alpha,s)$$
$$- \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m^-(\alpha,s).$$

There exist simple functions m_k, given by $m_k(\alpha,s) = \sum_{j=1}^{N_k} a_{kj}\chi_{A_{kj}}(\alpha,s)$, such that $m_k(\alpha,s) \to m^+(\alpha,s)$ as $k \to \infty$. These are determined by the partitions $0 = a_{k1} < a_{k2} < \ldots < a_{kN_k} = \delta(2M)$, and

$$A_{kj} = \{(\alpha,s)\colon a_{kj} \leq m^+(\alpha,s) < a_{k(j+1)}\},\ j = 1,\ldots,N_k-1,$$

$$A_{kN_k} = \{(\alpha,s)\colon m^+(\alpha,s) = a_{kN_k}\}.$$

By the Lebesgue Dominated Convergence Theorem,

$$\text{b)}\quad \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m^+(\alpha,s) = \lim_{k\to\infty} \Sigma_{j=1}^{N_k} a_{kj} \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]\chi_{A_{kj}}(\alpha,s).$$

For each $s \in [-M,M]$, $\beta \in [0,\delta(2M)]$, define $u(s;\beta) = \min[\{\alpha \in [-M,M]: m^+(\alpha,s) \geq \beta\} \cup \{M+1\}]$. Since $m^+(\cdot,s)$ is non-decreasing and right-continuous, $u(\cdot;\beta)$ is well defined. Fix β. For $a \in (-\infty,-M]$, $\{s: u(s;\beta) < a\}=\emptyset$, which is a Borel set. For $a \in (-M,M]$,

$$\{s:u(s;\beta) < a\} = \{s: \text{there exists } \alpha \in [-M,a) \text{ with } m^+(\alpha,s) \geq \beta\}$$

$$= \bigcup_{\alpha\in[-M,a)} \{s: m^+(\alpha,s) \geq \beta\} = \bigcup_{\alpha\in[-M,a)} S_\alpha(\beta).$$

Since m^+ is Borel-measurable, $\{(\alpha,s): m^+(\alpha,s) \geq \beta\}$ is a Borel set, and so its α-section, $S_\alpha(\beta)$, is also a Borel set. Assume $\sigma \in \bigcup_{\alpha\in[-M,a)} S_\alpha(\beta)$. Then there exists $\alpha' \in [-M,a)$ such that $\beta \leq m^+(\alpha',\sigma)$. If α' is irrational, since $m^+(\cdot,\sigma)$ is non-decreasing there exists rational $\alpha'' \in (\alpha',a)$ such that $\beta \leq m^+(\alpha'',\sigma)$. Thus, if $Ra = \{b \in R: b \text{ is rational}\}$,

$$\{s:u(s;\beta) < a\} = \bigcup_{\alpha\in[-M,a)} S_\alpha(\beta) = \bigcup_{\alpha\in[-M,a)\cap Ra} S_\alpha(\beta)$$

is a Borel set, since it is a countable union of Borel sets. Similarly, for $a \in (M,M+1]$,

$$\{s:u(s;\beta) < a\} = \bigcup_{\alpha\in[-M,M]} S_\alpha(\beta) = \bigcup_{\alpha\in[-M,M]\cap Ra} S_\alpha(\beta)$$

is a Borel set. For $a \in (M+1,\infty)$, $\{s: u(s;\beta) < a\} = [-M,M]$, which is a Borel set.

Since $\{s: u(s;\beta) < a\}$ is a Borel set for each $a \in R$, $u(\cdot;\beta)$ is Borel-measurable for each $\beta \in [0,\delta(2M)]$. From its definition, $u(s;\cdot)$ is non-decreasing for each $s \in [-M,M]$. Using the normalization $X(s,t) = 0$ for $s > t$, without loss of generality we may extend the domain of $X(\cdot,t)$ to $[-M,M+1]$, for each $t \in [-M,M]$. Hence, from b) we obtain

$$\text{c)} \quad \int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m^+(\alpha,s) = \lim_{k\to\infty} \{\Sigma_{j=1}^{N_k-1} a_{kj}[X(u(s,a_{k(j+1)}),t) - X(u(s,a_{kj}),t)] - a_{kN_k} X(u(s,a_{kN_k}),t)\}.$$

$X(u(s,a_{kj}),t)$ will be Borel-measurable in (s,t) if $(s,t) \to (u(s,\beta),t)$ is Borel-measurable. To check this, it is sufficient to look at $Q = \{(s,t): a_0 < u(s,\beta) < a_1,\ b_0 < t < b_1\}$ for all $a_0 < a_1$, $b_0 < b_1$. But $Q = u^{-1}((a_0,a_1),\beta) \times (b_0,b_1)$, and $u^{-1}((a_0,a_1),\beta)$ is a Borel set, hence Q is a Borel set. Thus $(s,t) \to (u(s,\beta),t)$, and also $(s,t) \to X(u(s,a_{kj}),t)$, is Borel-measurable. For each k, $\Sigma_{j=1}^{N_k-1} a_{kj}[X(u(s,a_{k(j+1)}),t) - X(u(s,a_{kj}),t)]$ is a finite sum of Borel-measurable functions, so it is Borel-measurable in (s,t). By c), $\int_{-M}^{M^+} d_\alpha[X(\alpha,t)]m^+(\alpha,s)$ is the limit of a sequence of Borel-measurable functions, hence it is Borel-measurable in (s,t). The same arguments hold with m^+ replaced by m^-, so by a) the proof is done.

Remark 2.3. The simplest type of function $\mu(t,\theta)$ satisfying (2.4), (2.5) and also of bounded variation in t is of the form $\mu(t,\theta) = \bar{\mu}(t-\theta)$. This form occurs in equations with constant lags and constant coefficients.

Lemma 2.4. If in lemma 2.1, for each θ, $\mu(\cdot,\theta)$ is also of bounded variation on every bounded set in R, then Y is Borel-measurable in (s,t).

Proof: Add to the induction hypothesis of the proof of lemma 2.1 that, where

it exists, Y is Borel-measurable in (s,t). This is true for $p = 0$ $(s \geq t)$, and holds for the function Y^o,

$$Y^o(s,t) = \begin{cases} Y(s,t) & s \geq t-p\varepsilon \\ Y(t-p\varepsilon,t) & s < t-p\varepsilon . \end{cases}$$

Then $Y^{k-1}(\alpha,t)\eta(\alpha,s)$ is Borel-measurable in (t,α,s); by [8 , theorem III. 11.13], $\int_{-M}^{M} Y^{k-1}(\alpha,t)\eta(\alpha,s)d\alpha$ is Borel-measurable in (s,t). By lemma 2.4, $\int_{-M}^{M^+} d_\alpha[Y^{k-1}(\alpha,t)]\mu(\alpha,s)$ is Borel-measurable in (s,t). Thus (rewriting by properties of Y, μ, and η)

$$Y^k(s,t) = E + \int_{-M}^{M^+} d_\alpha[Y^{k-1}(\alpha,t)]\mu(\alpha,s) - \int_{-M}^{M} Y(\alpha,t)\eta(\alpha,s)d\alpha$$

is Borel-measurable in (s,t). On the set $[-M,M] \times [-M,M] \cap \{(s,t): s \geq t-(p+1)\varepsilon\}$, $Y(s,t) = \lim_{k \to \infty} Y^k(s,t)$, so on that set $Y(s,t)$ is Borel-measurable. By induction and the arbitrary choice of M, the proof is complete.

Remark 2.4. Henry [15] makes the following definition:

> $F: R^2 \to R^{n^2}$ has property P if $F(t,s) = Q(t,s) + \sum_{s+\tau_\nu(t) \leq t} R_\nu(t)$ where the $\tau_\nu(\cdot)$, $R_\nu(\cdot)$ are continuous, $\sum_\nu |R_\nu(t)|$ converges uniformly; $Q(t,s)$ is continuous, absolutely continuous in s, with $\frac{\partial Q}{\partial s}(t,s)$ continuous in t for each s and $|\frac{\partial Q}{\partial s}(t,s)| \leq q(s)$ (all t) for some locally integrable $q(\cdot)$.

He then shows ([15], lemma 2) that if μ has property P, Y is Borel-measurable in (s,t). It is possible for F to have property P and not be of bounded variation in t on bounded sets. For example, if $q(s)$ is locally integrable, $r(t)$ is continuous but not of bounded variation on finite

intervals, then

$$F(t,s) = Q(t,s) = \begin{cases} 0 & s > t \\ -r(t)\int_s^t q(\sigma)d\sigma, & s \leq t \end{cases}$$

has property P but is not BV in t. The following are two examples of functions which are of bounded variation in t, but do not have property P:

$$F(t,s) = Q(t,s) = \begin{cases} 0\ , & s > t \\ \text{Cantor ternary function of } (t-s), & t-1 \leq s \leq t \\ 1\ , & s < t-1 \end{cases}$$

where Q is not absolutely continuous in s.

$$F(t,s) = Q(t,s) = \begin{cases} 0 & , \quad s > t-1 \\ (t-1-s) & , \quad t-2 < s \leq t-1 \\ 1 & , \quad s \leq t-2 \end{cases}$$

where Q is absolutely continuous in s, but $\frac{\partial Q}{\partial s}(t,s)$ is not continuous in t.

Lemma 2.5. Assume (2.4), (2.5), (2.8)-(2.11), and there exists a $\Delta > 0$ such that $\delta(\Delta) = 0$. Let $t_0 > \alpha_0 + \Delta$, and $\mu(t,\theta) = J(t,\theta) + \nu(t,\theta)$ be as in remark 2.1, where in addition $t < s$ implies $h_\ell(t) \leq h_\ell(s)$, and $\alpha_0 \leq h_\ell(s) < s - \Delta$ for all $s \in [t_0, a)$, $\ell = 1,\ldots,p$. Then $Y(s,\cdot)$ is right-continuous.

Proof: For $t < s$, $Y(s,t) = 0$; thus $Y(s,\cdot)$ is right-continuous on $(-\infty, s)$. Fix $T > s$. By lemma 2.2 and the assumptions above, for $t_0 \leq s \leq t$,

$$Y(s,t) = E + \sum_{\ell=1}^{p} a_\ell(t)Y(s,h_\ell(t)) + \int_s^t d_\theta[\nu(t,\theta)]Y(s,\theta)$$
$$+ \int_s^t \int_s^\lambda d_\theta[\eta(\lambda,\theta)]Y(s,\theta)d\lambda.$$

If $A_T = \sup\{|a_\ell(t)|: t \in [s,T],\ \ell = 1,\ldots,p\}$ and B_Y is the bound on Y given in lemma 2.1, for $t_0 \leq s \leq t \leq \tau \leq T$,

$$|Y(s,\tau)-Y(s,t)| \leq |\sum_{\ell=1}^{p} a_\ell(\tau)Y(s,h_\ell(\tau)) - \sum_{\ell=1}^{p} a_\ell(t)Y(s,h_\ell(t))|$$
$$+|\int_s^\tau d_\theta[\nu(\tau,\theta)]Y(s,\theta) - \int_s^t d_\theta[\nu(t,\theta)]Y(s,\theta)|$$
$$+ |\int_s^\tau \int_s^\lambda d_\theta[\eta(\lambda,\theta)]Y(s,\theta)d\lambda - \int_s^t \int_s^\lambda d_\theta[\eta(\lambda,\theta)]Y(s,\theta)d\lambda|$$
$$\leq \sum_{\ell=1}^{p} A_T|Y(s,h_\ell(\tau))-Y(s,h_\ell(t))| + \sum_{\ell=1}^{p} B_Y|a_\ell(\tau)-a_\ell(t)|$$
$$+ B_Y|\int_s^T d_\theta[\nu(\tau,\theta)-\nu(t,\theta)]| + B_Y \int_t^\tau m(\lambda)d\lambda.$$

Since the a_ℓ are continuous, the second term goes to zero as $\tau \to t$. By assumption b) of remark 2.1, the third term goes to zero as $\tau \to t$. Clearly the fourth term goes to zero as $\tau \to t$. Let $t \in [s,s+\Delta)$. Then, since $Y(s,\cdot)$ is right-continuous on $(-\infty,s)$, the h_ℓ are continuous and non-decreasing, and $h_\ell(\sigma) < \sigma-\Delta$ for $\sigma \geq t_0$, the first term goes to zero as $\tau \to t$. Hence $Y(s,\cdot)$ is right-continuous on $(-\infty,s+\Delta)$. We continue in this manner, showing $Y(s,\cdot)$ is right-continuous on $(-\infty,s+j\Delta)$ for $j = 1,2,\ldots,N_0$, where N_0 = greatest integer in $[\frac{T-s}{\Delta}]$. Taking one more step, for $t \in [s+N_0\Delta,T)$, we have that $Y(s,\cdot)$ is right continuous on $(-\infty,T)$. Since this holds for each $T > s$, the lemma is proved.

Lemma 2.6. Assume (2.4), (2.5), (2.8)-(2.11), and there exists a $\Delta > 0$ such that $\delta(\Delta) = 0$. Let $t_0 > \alpha_0+\Delta$, and $\mu(t,\theta) = J(t,\theta) + \nu(t,\theta)$ be as in remark 2.1, where in addition $t_0 \leq t < s$ implies $h_\ell(t) < h_\ell(s)$, and $\alpha_0 \leq h_\ell(s) < s-\Delta$ for all $s \in [t_0,a)$, $\ell = 1,\ldots,p$. Assume that for all $w \in BV([\alpha_0,a),R^n)$, $t \in [t_0,a)$, the map $s \rightarrow \int_s^t w(\alpha)\eta(\alpha,s)d\alpha$ is continuous on $[t_0,a)$. Then, if $s,t \in [t_0,a)$, $Y(s,t^-) = Y(s^+,t)$.

Proof: Since the h_ℓ are strictly monotone, the h_ℓ^{-1} are well defined and continuous, $\ell= 1,\ldots,p$. Thus, by the representation of μ, (2.9) becomes

$$(2.15) \qquad Y(s,t) = E - \Sigma_{\ell=1}^{p} \int_{[s,t+\Delta]\cap h_\ell^{-1}[s,\infty)} d_\alpha[Y(\alpha,t)]a_\ell(\alpha) + \int_s^{t^+} d_\alpha[Y(\alpha,t)]\nu(\alpha,s) - \int_s^t Y(\alpha,t)\eta(\alpha,s)d\alpha.$$

Since $\nu(\alpha,\cdot)$ is continuous and $\nu(s,s) = 0$, for each $t \in [t_0,a)$, $\int_s^{t^+} d_\alpha[Y(\alpha,t)]\nu(\alpha,s)$ is continuous in s. We have assumed that, for each $t \in [t_0,a)$, $\int_s^t Y(\alpha,t)\eta(\alpha,s)d\alpha$ is continuous in s. Thus there is a possible discontinuity of $Y(\cdot,t)$ at any

$$s \in H_1(t) = \{h_\ell(t): \ell = 1,\ldots,p\},$$

at any

$$s \in H_2(t) = \{h_j(h_\ell(t)): j,\ell = 1,\ldots,p\},$$

at any

$$s \in H_3(t) = \{h_i(h_j(h_\ell(t))): i,j,\ell = 1,\ldots,p\}, \text{ etc.}$$

Since $h_\ell(\tau) < \tau-\Delta$, there are a finite number of discontinuities on any finite interval $[\tau,t]$. By (2.15) a discontinuity of $Y(\cdot,t)$ at $s < t$ is given by

$$(2.16) \qquad Y(s,t)-Y(s^+,t) = \Sigma_{\ell=1}^{p}[Y(h_\ell^{-1}(s),t) - Y(h_\ell^{-1}(s)^+,t)]a_\ell(h_\ell^{-1}(s)).$$

By Lemma 2.2, for $t \geq s \geq t_0$,

$$(2.17) \qquad Y(s,t) = E + \Sigma_{\ell=1}^{p} a_\ell(t)Y(s,h_\ell(t)) + \int_s^t d_\theta[\nu(t,\theta)]Y(s,\theta) + \int_s^t \int_s^\lambda d_\theta[\eta(\lambda,\theta)]Y(s,\theta)d\lambda.$$

By assumption b) of Remark 2.1, $\int_s^t d_\theta[\nu(t,\theta)]Y(s,\theta)$ is continuous in t; clearly the last term of (2.17) is continuous in t. Hence there is a possible discontinuity of $Y(s,\cdot)$ at those t such that $s \in H_i(t)$ for some $i \in \{0,1,2,3\ldots\}$. Also, a discontinuity of $Y(s,\cdot)$ at $t > s$ is given by

$$Y(s,t) - Y(s,t^-) = \textstyle\sum_{\ell=1}^{p} a_\ell(t)[Y(s,h_\ell(t)) - Y(s,h_\ell(t)^-)]. \tag{2.18}$$

Fix t and s in $[t_0,a)$, with $t > s$. Then $Y(s,\cdot)$ and $Y(\cdot,t)$ have the same possible points of discontinuity in $[s,t]$; order them $t = \tau_0 > \tau_1 > \cdots > \tau_q = s$. The discontinuities in $Y(\cdot,t)$ are:

$$Y(\tau_0,t) - Y(\tau_0^+,t) = E.$$

$$Y(\tau_1,t) - Y(\tau_1^+,t) = \sum_{\{\ell:\ h_\ell^{-1}(\tau_1) = \tau_0\}} a_\ell(\tau_0).$$

$$Y(\tau_2,t) - Y(\tau_2^+,t) = \sum_{\{\ell:\ h_\ell^{-1}(\tau_2) = \tau_1\}} \Big[\sum_{\{j:\ h_j^{-1}(\tau_1) = \tau_0\}} a_j(\tau_0)\Big] a_\ell(\tau_1) + \sum_{\{\ell:\ h_\ell^{-1}(\tau_2) = \tau_0\}} a_\ell(\tau_0).$$

$$\begin{aligned} Y(s,t) - Y(s^+,t) = {} & \sum_{\{\ell:\ h_\ell^{-1}(\tau_q) = \tau_{q-1}\}} \Big[\sum_{\{j:\ h_j^{-1}(\tau_{q-1}) = \tau_{q-2}\}} \Big\{\cdots \\ & \cdots\Big[\sum_{\{i:\ h_i^{-1}(\tau_1) = \tau_0\}} a_i(\tau_0)\Big]\cdots\Big\} a_j(\tau_{q-2})\Big] a_\ell(\tau_{q-1}) \\ & + \cdots + \sum_{\{\ell:\ h_\ell^{-1}(\tau_q) = \tau_0\}} a_\ell(\tau_0). \end{aligned}$$

The discontinuities in $Y(s,\cdot)$ are:

$$Y(s,\tau_q) - Y(s,\tau_q^-) = E.$$

$$Y(s,\tau_{q-1}) - Y(s,\tau_{q-1}^-) = \sum_{\{\ell:\ h_\ell(\tau_{q-1}) = \tau_q\}} a_\ell(\tau_{q-1}).$$

$$\begin{aligned} Y(s,\tau_{q-2}) - Y(s,\tau_{q-2}^-) = & \sum_{\{\ell:\ h_\ell(\tau_{q-2}) = \tau_{q-1}\}} a_\ell(\tau_{q-2}) \cdot \\ & \qquad [\sum_{\{j:\ h_j(\tau_{q-1})=\tau_q\}} a_j(\tau_{q-1})] \\ & \qquad\quad + \sum_{\{\ell:\ h_\ell(\tau_{q-2}) = \tau_q\}} a_\ell(\tau_{q-2}). \end{aligned}$$

$$\begin{aligned} Y(s,t) - Y(s,t^-) = & \sum_{\{\ell:\ h_\ell(\tau_0) = \tau_1\}} a_\ell(\tau_0)[\sum_{\{j:h_j(\tau_1)=\tau_2\}} a_j(\tau_1)\{\cdots \\ & \cdots[\sum_{\{i:\ h_i(\tau_{q-1}) = \tau_q\}} a_i(\tau_{q-1})]\cdots\}] \\ & + \cdots + \sum_{\{\ell:\ h_\ell(\tau_0) = \tau_q\}} a_\ell(\tau_0). \end{aligned}$$

By interchanging the order of summation in one of the general expressions, we see that $Y(s,t) - Y(s,t^-) = Y(s,t) - Y(s^+,t)$, and so $Y(s,t^-) = Y(s^+,t)$.

Remark 2.5. For the map $s \to \int_s^t w(\alpha)\eta(\alpha,s)d\alpha$ to be continuous on $[t_0,a)$ for each $w \in BV([\alpha_0,a),R^n)$ and $t \in [t_0,a)$, it is sufficient that in addition to (2.8),

$$\eta(t,\theta) = b_0(t)\zeta(t,\theta) + \Sigma_{\ell=1}^q b_\ell(t)X[g_\ell(t)](\theta) + \bar{\eta}(t,\theta)$$

where $b_\ell: [t_0,a) \to R^{n^2}$ is integrable, $\ell = 0,\ldots,q$;

$$\zeta(t,\theta) = \begin{cases} -E, & \theta < t \\ 0, & t \leq \theta \end{cases} \quad \text{for } (t,\theta) \in R^2;$$

χ as in remark 2.1; g_ℓ is continuous and strictly increasing on $[t_0,a)$, $\alpha_0 < g_\ell(t) < t$ for $t \in [t_0,a)$, $\ell = 1,\ldots,q$; $\bar{\eta}(t,\cdot)$ is continuous on $[\alpha_0,a)$ for all $t \in [t_0,a)$.

We now consider a set of conditions under which solutions exist within the class $BV([\alpha_0,a),R^n)$. For the proof it is useful to note that

$$\begin{aligned} &|\Sigma_{\ell=1}^p a_\ell(t)x(h_\ell(t)) - \Sigma_{\ell=1}^p a_\ell(s)x(h_\ell(s))| \\ &\quad \leq \Sigma_{\ell=1}^p |a_\ell(t)-a_\ell(s)|\,\|x\|_{\max\{t,s\}} + \Sigma_{\ell=1}^p |a_\ell(s)|\,|x(h_\ell(t))-x(h_\ell(s))|, \end{aligned}$$

where $\|x\|_\sigma = \sup\limits_{\tau \in [\alpha_0,\sigma]} |x(\tau)|$. Thus,

$$\begin{aligned} \operatorname*{var}_{[t_1,t_2]} \Sigma_{\ell=1}^p a_\ell(\cdot)x(h_\ell(\cdot)) &\leq \|x\|_{t_2} \Sigma_{\ell=1}^p \operatorname*{var}_{[t_1,t_2]} a_\ell \\ &\quad + \Sigma_{\ell=1}^p \sup_{s \in [t_1,t_2]} |a_\ell(s)| \operatorname*{var}_{[h_\ell(t_1),h_\ell(t_2)]} x \end{aligned}$$

under suitable assumptions on the a_ℓ and h_ℓ, as below.

Lemma 2.7. Assume (2.4), (2.5), (2.8), and there exists a $\Delta > 0$ such that $\delta(\Delta) = 0$. Let $t_0 > \alpha_0 + \Delta$, and $\mu(t,\theta) = J(t,\theta) + \nu(t,\theta)$ be as in remark 2.1, where in addition $t < s$ implies $h_\ell(t) \leq h_\ell(s)$, and $\alpha_0 \leq h_\ell(s) < s-\Delta$ for all $s \in [t_0,a)$, $\ell = 1,\ldots,p$; the a_ℓ are of bounded variation on every bounded interval; and there exists $L > 0$ such that

$$\int_{\alpha_0}^{\infty} d_\theta |\nu(t,\theta) - \nu(s,\theta)| \leq L|t-s| .$$

Then there exists in $BV([\alpha_0,T],R^n)$ a solution to

$$x_{t_0} = \varphi,$$

$$x(t) = \varphi(t_0) - \int_{\alpha_0}^{t_0} d_\theta[\mu(t_0,\theta)]\varphi(\theta) + \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]x(\theta)$$

$$+ \int_{t_0}^{t} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x(\theta)ds \quad \text{on} \quad [t_0,T]$$

where $\varphi \in BV([\alpha_0,t_0],R^n)$, φ is right-continuous, and $t_0 < T < a$.

Proof: For $k = 1,2,3,\ldots$ define $\tau_k = \frac{T-t_0}{k}$, and the sequence of functions

$$(2.19)\quad x_k(t) = \begin{cases} \varphi(t), & t \in [\alpha_0,t_0+\tau_k] \\ \varphi(t_0) - \int_{\alpha_0}^{t_0} d_\theta[\mu(t_0,\theta)]\varphi(\theta) + \int_{\alpha_0}^{t-\tau_k} d_\theta[\mu(t,\theta)]x_k(\theta) & \\ \quad + \int_{t_0}^{t-\tau_k}\int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x_k(\theta)ds, & t \in (t_0+\tau_k,T]. \end{cases}$$

x_k is right-continuous on $[\alpha_0,t_0+\tau_k]$ since φ is; stepping over intervals of size τ_k, the continuity of $g(\psi,s)$ expressed in (2.4) and the right-continuity of x_k on previous intervals yields that, so long as x_k exists on $[\alpha_0,T]$, it is right-continuous.

For $t \in [t_0,t_0+\Delta]$, since $\delta(\Delta) = 0$,

$$|x_k(t)| \leq \|\varphi\| + 2\delta(T-\alpha_0)\|\varphi\| + \int_{t_0}^{t-\tau_k} m(s)\|x_k\|_s ds .$$

Since the right-hand side is non-decreasing in t,

$$\|x_k\|_t \leq [1 + 2\delta(T-\alpha_0)]\|\varphi\| + \int_{t_0}^{t-\tau_k} m(s)\|x_k\|_s ds .$$

By Gronwall's inequality,

$$\|x_k\|_t \leq [1 + 2\delta(T-\alpha_0)]\|\varphi\| \exp\{\int_{t_0}^{t-\tau_k} m(s)ds\}.$$

Similarly, on $[t_0+\Delta, t_0+2\Delta]$, since the second part of (2.19) can be rewritten

$$x_k(t) = x_k(t_0+\Delta) - \int_{\alpha_0}^{t_0+\Delta-\tau_k} d_\theta[\mu(t_0+\Delta,\theta)]x_k(\theta) + \int_{\alpha_0}^{t-\tau_k} d_\theta[\mu(t,\theta)]x_k(\theta)$$
$$+ \int_{t_0+\Delta-\tau_k}^{t-\tau_k} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x_k(\theta)ds \quad \text{if} \quad \tau_k < \Delta,$$

we have

$$|x_k(t)| \leq \|x_k\|_{t_0+\Delta} + 2\delta(T-\alpha_0)\|x_k\|_{t_0+\Delta} + \int_{t_0+\Delta-\tau_k}^{t-\tau_k} m(s)\|x_k\|_s ds ,$$

$$\|x_k\|_t \leq [1 + 2\delta(T-\alpha_0)]\|x_k\|_{t_0+\Delta} + \int_{t_0+\Delta-\tau_k}^{t-\tau_k} m(s)\|x_k\|_s ds ,$$

$$\|x_k\|_t \leq [1 + 2\delta(T-\alpha_0)]\|x_k\|_{t_0+\Delta} \exp\{\int_{t_0+\Delta-\tau_k}^{t-\tau_k} m(s)ds\}$$

$$\leq [1 + 2\delta(T-\alpha_0)]^2\|\varphi\| \exp\{\int_{t_0}^{t-\tau_k} m(s)ds\} .$$

Now, if $\nu_0 = 1 +$ greatest integer in $[\frac{T-t_0}{\Delta}]$, for all k we have by induction

$$\|x_k\|_T \leq [1 + 2\delta(T-\alpha_0)]^{\nu_0}\|\varphi\| \exp\{\int_{t_0}^{T} m(s)ds\} = B_x,$$

a uniform bound on the x_k, as long as they exist over $[\alpha_0, T]$.

For $k \geq \nu_0$ (so that $\tau_k < \Delta$), we re-write (2.19),

$$x_k(t) = \varphi(t_0) - \int_{\alpha_0}^{t_0} d_\theta[\mu(t_0,\theta)]\varphi(\theta) + \Sigma_{\ell=1}^{p} a_\ell(t)x_k(h_\ell(t))$$
$$+ \int_{\alpha_0}^{t} d_\theta[\nu(t,\theta)]x_k(\theta) + \int_{t_0}^{t-\tau_k} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x_k(\theta)ds,$$
$$t \in (t_0+\tau_k, T].$$

Then

$$\operatorname*{var}_{[\alpha_0,t_0]} x_k = \operatorname*{var}_{[\alpha_0,t_0]} \varphi.$$

$$\operatorname*{var}_{[\alpha_0,t_0+\Delta]} x_k \leq \operatorname*{var}_{[\alpha_0,t_0]} \varphi + \|x_k\|_T \Sigma_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+\Delta]} a_\ell$$
$$+ \operatorname*{var}_{[\alpha_0,t_0]} \varphi \,\Sigma_{\ell=1}^{p} \sup_{s\in[t_0,t_0+\Delta]} |a_\ell(s)|$$
$$+ L\Delta\|x_k\|_T + \int_{t_0}^{t_0+\Delta-\tau_k} m(s)\|x_k\|_s ds$$
$$\leq [1 + \Sigma_{\ell=1}^{p} \sup_{s\in[t_0,t_0+\Delta]} |a_\ell(s)|] \operatorname*{var}_{[\alpha_0,t_0]} \varphi$$
$$+ [\Sigma_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+\Delta]} a_\ell + L\Delta + \int_{t_0}^{t_0+\Delta-\tau_k} m(s)ds]B_x.$$

Re-writing (2.19) again,

$$\begin{aligned} x_k(t) = {} & x_k(t_0+\Delta) - \int_{\alpha_0}^{t_0+\Delta} d_\theta[\mu(t_0+\Delta,\theta)]x_k(\theta) + \sum_{\ell=1}^{p} a_\ell(t)x_k(h_\ell(t)) \\ & + \int_{\alpha_0}^{t} d_\theta[\nu(t,\theta)]x_k(\theta) + \int_{t_0+\Delta-\tau_k}^{t-\tau_k} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x_k(\theta)ds, \end{aligned}$$

$$t \in (t_0+\Delta, T].$$

Thus

$$\begin{aligned} \operatorname*{var}_{[\alpha_0,t_0+2\Delta]} x_k \leq {} & \operatorname*{var}_{[\alpha_0,t_0+\Delta]} x_k + \|x_k\|_T \sum_{\ell=1}^{p} \operatorname*{var}_{[t_0+\Delta,t_0+2\Delta]} a_\ell \\ & + \operatorname*{var}_{[\alpha_0,t_0+\Delta]} x_k \sum_{\ell=1}^{p} \sup_{s\in[t_0+\Delta,t_0+2\Delta]} |a_\ell(s)| \\ & + L\Delta\|x_k\|_T + \int_{t_0+\Delta-\tau_k}^{t_0+2\Delta-\tau_k} m(s)\|x_k\|_s ds \\ \leq {} & [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+\Delta]} |a_\ell(s)|] \operatorname*{var}_{[\alpha_0,t_0]} \varphi \\ & + [\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+2\Delta]} a_\ell + 2L\Delta + \int_{t_0}^{t_0+2\Delta-\tau_k} m(s)ds]B_x \\ & + \sum_{\ell=1}^{p} \sup_{s\in[t_0+\Delta,t_0+2\Delta]} |a_\ell(s)| \{[1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+\Delta]} |a_\ell(s)|] \cdot \\ & \qquad \operatorname*{var}_{[\alpha_0,t_0]} \varphi \\ & + [\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+\Delta]} a_\ell + L\Delta + \int_{t_0}^{t_0+\Delta-\tau_k} m(s)ds]B_x\} \\ \leq {} & [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} |a_\ell(s)|]^2 \operatorname*{var}_{[\alpha_0,t_0]} \varphi \\ & + [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} |a_\ell(s)|][\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+2\Delta]} a_\ell \end{aligned}$$

$$+ 2L\Delta + \int_{t_0}^{t_0+2\Delta-\tau_k} m(s)ds]B_x.$$

With the equation re-written to begin with $x_k(t_0+2\Delta)$, we have

$$\operatorname*{var}_{[\alpha_0,t_0+3\Delta]} x_k \leq \operatorname*{var}_{[\alpha_0,t_0+2\Delta]} x_k + \|x_k\|_T \sum_{\ell=1}^{p} \operatorname*{var}_{[t_0+2\Delta,t_0+3\Delta]} a_\ell$$

$$+ \operatorname*{var}_{[\alpha_0,t_0+2\Delta]} x_k \sum_{\ell=1}^{p} \sup_{s\in[t_0+2\Delta,t_0+3\Delta]} |a_\ell(s)|$$

$$+ L\Delta\|x_k\|_T + \int_{t_0+2\Delta-\tau_k}^{t_0+3\Delta-\tau_k} m(s)\|x_k\|_s ds$$

$$\leq [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} |a_\ell(s)|]^2 \operatorname*{var}_{[\alpha_0,t_0]} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} |a_\ell(s)|][\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+2\Delta]} a_\ell$$

$$+ 2L\Delta + \int_{t_0}^{t_0+2\Delta-\tau_k} m(s)ds]B_x$$

$$+ [\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0+2\Delta,t_0+3\Delta]} a_\ell + L\Delta + \int_{t_0+2\Delta-\tau_k}^{t_0+3\Delta-\tau_k} m(s)ds]B_x$$

$$+ \sum_{\ell=1}^{p} \sup_{s\in[t_0+2\Delta,t_0+3\Delta]} |a_\ell(s)| \{[1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} \cdot$$

$$|a\ (s)|]^2 \operatorname*{var}_{[\alpha_0,t_0]} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \sup_{s\in[t_0,t_0+2\Delta]} |a_\ell(s)|][\sum_{\ell=1}^{p} \operatorname*{var}_{[t_0,t_0+2\Delta]} a_\ell$$

$$+ 2L\Delta + \int_{t_0}^{t_0+2\Delta-\tau_k} m(s)ds]B_x\}$$

$$\leq [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|]^2 \underset{[\alpha_0,t_0]}{\operatorname{var}} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|][\sum_{\ell=1}^{p} \underset{[t_0,t_0+3\Delta]}{\operatorname{var}} a_\ell$$

$$+ 3L\Delta + \int_{t_0}^{t_0+3\Delta-\tau_k} m(s)ds]B_x$$

$$+ \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|\{[1+ \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|]^2 \underset{[\alpha_0,t_0]}{\operatorname{var}} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|][\sum_{\ell=1}^{p} \underset{[t_0,t_0+3\Delta]}{\operatorname{var}} a_\ell$$

$$+ 3L\Delta + \int_{t_0}^{t_0+3\Delta-\tau_k} m(s)ds]B_x\}$$

$$= [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|]^3 \underset{[\alpha_0,t_0]}{\operatorname{var}} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,t_0+3\Delta]}{\sup} |a_\ell(s)|]^2[\sum_{\ell=1}^{p} \underset{[t_0,t_0+3\Delta]}{\operatorname{var}} a_\ell$$

$$+ 3L\Delta + \int_{t_0}^{t_0+3\Delta-\tau_k} m(s)ds]B_x.$$

Continuing in this manner, for $k \geq \nu_0$,

$$\underset{[\alpha_0,T]}{\operatorname{var}} x_k \leq [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,T]}{\sup} |a_\ell(s)|]^{\nu_0} \underset{[\alpha_0,t_0]}{\operatorname{var}} \varphi$$

$$+ [1 + \sum_{\ell=1}^{p} \underset{s\in[t_0,T]}{\sup} |a_\ell(s)|]^{\nu_0-1} [\sum_{\ell=1}^{p} \underset{[t_0,T]}{\operatorname{var}} a_\ell$$

$$+ \nu_0 L\Delta + \int_{t_0}^{T} m(s)ds]B_x.$$

Thus $\operatorname*{var}_{[\alpha_0,t]} x_k$ is uniformly bounded in $k = \nu_0, \nu_0+1, \ldots,$ for all $t \in [t_0,T]$ for which $x_k(t)$ exists. But since each of the x_k is of bounded variation, right-continuous, and bounded by a constant for as long as it exists on $[\alpha_0,T]$, the defining relation (2.19) shows that x_k exists over all of $[\alpha_0,T]$ for $k = \nu_0, \nu_0+1, \ldots$. By Helly's theorem, there is a subsequence, which we shall also denote by x_k, and an $x \in BV([\alpha_0,T],R^n)$ such that

$$x_k(t) \to x(t) \quad \text{as} \quad k \to \infty, \text{ for all } t \in [\alpha_0,T].$$

By the Dominated Convergence Theorem, and the facts that

$$\int_{t-\tau_k}^{t} d_\theta|\mu(t,\theta)| \to 0 \quad \text{as} \quad k \to \infty, \text{ and}$$

$$\int_{t-\tau_k}^{t} m(s)ds \to 0 \quad \text{as} \quad k \to \infty, \text{ we have that, for } t \in [t_0,T],$$

$$\int_{\alpha_0}^{t-\tau_k} d_\theta[\mu(t,\theta)]x_k(\theta) \to \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]x(\theta) \quad \text{as} \quad k \to \infty, \text{ and}$$

$$\int_{t_0}^{t-\tau_k} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x_k(\theta)ds \to \int_{t_0}^{t} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x(\theta)ds$$

as $k \to \infty$. Thus

$$x(t) = \begin{cases} \varphi(t), & t \in [\alpha_0,t_0] \\ \varphi(t_0) - \int_{\alpha_0}^{t_0} d_\theta[\mu(t_0,\theta)]\varphi(\theta) + \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]x(\theta) & \\ \qquad + \int_{t_0}^{t} \int_{\alpha_0}^{s} d_\theta[\eta(s,\theta)]x(\theta)ds, & t \in [t_0,T], \end{cases}$$

and x is the desired solution.

3. Quasiconvex Families of Functions

We now present definitions and resulting properties which will form the basis of the proof of the necessary conditions. Let

$I = [\alpha_0, a)$ be a bounded interval containing $[\alpha_0, t_0]$,

$I' = (t_0, a)$,

$\mathcal{G}$ be a fixed, open, convex region in R^n (possibly all of R^n),

$\mathcal{F}$ be a family of functions $F\colon C(I,\mathcal{G}) \times I' \to R^n$,

$P^k = \{\alpha \in R^k\colon \alpha^i \geq 0$ for $i = 1,\ldots,k$ and $\sum_{i=1}^k \alpha^i = 1\}$, k any positive integer.

Definition 3.1. For compact $\Psi \subset C([\alpha_0, t_0], R^n)$, non-negative $K \in L^1(I', R)$, $X \subset \mathcal{G}$, let

$$C(I,X)_{\Psi,K} = \{x \in C(I,X)\colon \left|\frac{d}{dt} D(x(\cdot),t)\right| \leq K(t) \text{ a.e. on } I',$$

$$x_{t_0} \in \Psi\}.$$

Definition 3.2. A family $\mathcal{F}$ shall be called quasi-convex if:

a) Each $F(x(\cdot),t) \in \mathcal{F}$ is C^1 in x for fixed $t \in I'$ and measurable on I' for fixed $x \in C(I,\mathcal{G})$.

b) Given any $F \in \mathcal{F}$ and any compact, convex $X \subset \mathcal{G}$, there exists $m \in L^1(I',R)$ [m depending on X,F] so that, for all $t \in I'$ and $x \in C(I,X)$, $|F(x(\cdot),t)| \leq m(t)$ and $\|dF[x(\cdot),t;\ .]\| \leq m(t)$ [i.e. $|dF[x(\cdot),t;\psi]| \leq m(t)\|\psi\|_t$ for $\psi \in C(I,R^n)$], where dF is the Frechet differential of F with respect to x.

c) For every compact, convex X contained in $\mathcal{G}$, compact $\Psi \subset$

$C([\alpha_0,t_0],X)$, non-negative $K \in L^1(I',R)$, finite collection $\{F_1,\ldots,F_k\} \subset \mathscr{F}$, and $\varepsilon > 0$: there exists for each $\beta \in P^k$ an $F_\beta \in \mathscr{F}$, [F_β depending on X,Ψ,K, the F_i, and ε], satisfying

$$|F_\beta(x(\cdot),t)| \leq \Sigma_{i=1}^k m_i(t), \quad \|dF_\beta[x(\cdot),t;\cdot]\| \leq \Sigma_{i=1}^k m_i(t)$$

for each $\beta \in P^k$, $t \in I'$, and $x \in C(I,X)$ [where the m_i are the $L^1(I',R)$ functions described in b), depending on X and the F_i], so that

$$G(x(\cdot),t;\beta) = \Sigma_{i=1}^k \beta^i F_i(x(\cdot),t) - F_\beta(x(\cdot),t)$$

satisfies

(c') $\left|\int_{\tau_1}^{\tau_2} G(x(\cdot),t;\beta)dt\right| < \varepsilon$ for all $\beta \in P^k$, $[\tau_1,\tau_2] \subset I'$, and $x \in C(I,X)_{\Psi,K}$,

(c") if $\{\beta_i\}_{i=1}^\infty$ is a sequence in P^k, so that $\beta_i \to \bar{\beta} \in P^k$, then $\{G(x(\cdot),t;\beta_i)\}_{i=1}^\infty$ converges in measure on I' to $G(x(\cdot),t;\bar{\beta})$ for each $x \in C(I,X)_{\Psi,K}$.

Remark 3.1. By the definition of $G(x(\cdot),t;\beta)$,

$$|G(x(\cdot),t;\beta)| \leq 2\ \Sigma_{i=1}^k m_i(t) \quad \text{and}$$

$$\|dG[x(\cdot),t;\beta;\cdot]\| \leq 2\ \Sigma_{i=1}^k m_i(t)$$

for all $x \in C(I,X)$, $\beta \in P^k$, and $t \in I'$.

Remark 3.2. If $\mathscr{F}$ is convex, then it is also quasiconvex, with $F_\beta = \Sigma_{i=1}^k \beta^i F_i$.

Lemma 3.1. Let $X \subset R^n$, $\Psi \subset C([\alpha_0,t_0],X)$, $K \in L^1(I',R)$. Assume $K \geq 0$, X and Ψ compact, (2.3) - (2.5), and

(3.1) there exists $\lambda > 0$, $L > 0$ such that, if $s \leq t$,

$$\int_{t-\lambda}^{t} |d_\theta[\mu(t,\theta) - \mu(s,\theta)]| \leq L|t-s|.$$

Then $C(I,X)_{\Psi,K}$ is a compact subset of $C(I,R^n)$. If X and Ψ are convex, $C(I,X)_{\Psi,K}$ is convex.

Proof: Convexity follows from convexity of Ψ and K and linearity of D. $C(I,X)$ is bounded, since X is compact. Let $x \in C(I,X)_{\Psi,K}$. Then

$$\begin{aligned} |x(t)-x(t_0)| &\leq |D(x(\cdot),t) - D(x(\cdot),t_0)| \\ &\quad + |g(x(\cdot),t)-g(x_{t_0},t)| + |g(x_{t_0},t)-g(x_{t_0},t_0)| \\ &\leq \int_{t_0}^{t} K(s)ds + \left|\int_{t_0}^{t} d_\theta[\mu(t,\theta)][x(\theta)-x(t_0)]\right| \\ &\quad + |g(x_{t_0},t)-g(x_{t_0},t_0)|. \end{aligned}$$

Choose γ, $0 < \gamma \leq \lambda$, such that $\delta(\gamma) < 1$. Let $b = [1-\delta(\gamma)]^{-1}$. Then if $t \in [t_0,t_0+\gamma]$,

$$\begin{aligned} |x(t)-x(t_0)| &\leq \delta(\gamma)\|x-x_{t_0}\|_t + |g(x_{t_0},t)-g(x_{t_0},t_0)| \\ &\quad + \int_{t_0}^{t} K(s)ds \\ &\leq \delta(\gamma)\|x-x_{t_0}\|_t + \sup_{\substack{t\in[t_0,t_0+\gamma] \\ \psi \in \Psi}} |g(\psi,t)-g(\psi,t_0)| \end{aligned}$$

$$+ \int_{t_0}^{t} K(s)ds.$$

Since the right hand side is non-decreasing in t,

$$\|x-x_{t_0}\|_t \leq \delta(\gamma)\|x-x_{t_0}\|_t + \sup_{\substack{t\in[t_0,t_0+\gamma]\\ \psi\in\Psi}} |g(\psi,t)-g(\psi,t_0)|$$

$$+ \int_{t_0}^{t} K(s)ds.$$

$$\|x-x_{t_0}\|_t \leq b \sup_{\substack{t\in[t_0,t_0+\gamma]\\ \psi\in\Psi}} |g(\psi,t)-g(\psi,t_0)| + b\int_{t_0}^{t} K(s)ds.$$

The supremum is finite since g is continuous and the supremum is taken over a compact set. For $t_0 \leq t' \leq t \leq t_0+\gamma$,

$$|x(t)-x(t')| \leq |D(x(\cdot),t)-D(x(\cdot),t')| + |g(x_{t_0},t)-g(x_{t_0},t')|$$

$$+ \left|\int_{t_0}^{t} d_\theta[\mu(t,\theta)-\mu(t',\theta)][x(\theta)-x(t_0)]\right|$$

$$\leq \int_{t'}^{t} K(s)ds + \sup_{\psi\in\Psi} |g(\psi,t)-g(\psi,t')|$$

$$+ bL|t-t'|\Big[\sup_{\substack{t\in[t_0,t_0+\gamma]\\ \psi\in\Psi}} |g(\psi,t)-g(\psi,t_0)| + \int_{t_0}^{t_0+\gamma} K(s)ds\Big].$$

By the continuity of g and compactness of Ψ, the elements of $C(I,X)_{\Psi,K}$ are equicontinuous on $[\alpha_0,t_0+\gamma]$. By the Arzela-Ascoli theorem, $C(I,X)_{\Psi,K}$ restricted to $[\alpha_0,t_0+\gamma]$ is contained in compact $\Psi_1 \subset C([\alpha_0,t_0+\gamma],X)$.

By the same arguments, we show that $C(I,X)_{\Psi,K}$ is equicontinuous, and hence conditionally compact, when its elements are restricted to $[\alpha_0,t_0+2\gamma]$, $[\alpha_0,t_0+3\gamma],\ldots,[\alpha_0,t_0+N\gamma]$, where N = greatest integer in $\frac{a-t_0}{\gamma}$. Then, as before, we obtain equicontinuity on the interval $[\alpha_0,a)$, (noting that g is defined for $t = a$, and including this value in the supremums). By [8, theorem IV.6.5], $C(I,X)_{\Psi,K}$ is conditionally compact.

Let $x^j \in C(I,X)_{\Psi,K}$, $j = 1,2,\ldots$, $x^j \to x$ as $j \to \infty$. Since X is compact, $x \in C(I,X)$. Ψ compact and $x^j_{t_0} \in \Psi$, $j = 1,2,\ldots$ imply that $x_{t_0} \in \Psi$. $\left|\frac{d}{dt} D(x^j(\cdot),t)\right| \leq K(t)$, $j = 1,2,\ldots$, a.e. on I' and D continuous imply that $\left|\frac{d}{dt} D(x(\cdot),t)\right| \leq K(t)$ a.e. on I'. Thus $x \in C(I,X)_{\Psi,K}$, so $C(I,X)_{\Psi,K}$ is closed, hence compact.

Remark 3.3. As an example of a $\mu(t,\theta)$ which satisfies (3.1), let

$$\mu(t,\theta) = \Sigma_{\ell=1}^{p} a_\ell(t)\chi[h_\ell(t)](\theta) + \nu(t,\theta)$$

as in Remark 2.1. In addition, assume that there exist $\lambda > 0$, $L > 0$ such that

i) $\alpha_0 \leq h_\ell(s) < s-\lambda$, $\ell = 1,\ldots,p$, $s \in [t_0,a)$,

ii) if $t_0 \leq s \leq t < a$, $\int_{t-\lambda}^{t} d_\theta|\nu(t,\theta)-\nu(s,\theta)| \leq L|t-s|$.

Then clearly (3.1) is satisfied.

Lemma 3.2. Assume the conditions of lemma 3.1 hold, and $\varepsilon > 0$. Let $F_j(x(\cdot),t)$, $j = 1,\ldots,k$, be mappings from $C(I,X) \times I'$ into R^n which are measurable in t for fixed x, and C^1 in x for fixed t. Assume there exists $m \in L^1(I',R)$ such that

$$|F_j(x(\cdot),t)| \leq m(t), \quad \|dF_j[x(\cdot),t;\cdot]\| \leq m(t)$$

for all $x \in C(I,X)$, $t \in I'$, $j = 1,\dots,k$. Let $p_j(t)$, $j = 1,\dots,k$, be given non-negative, real valued, measurable functions satisfying $\sum_{j=1}^{k} p_j(t) = 1$ a.e. on I'. Then it is possible to subdivide I' into sufficiently small disjoint sub-intervals I_i, $i = \pm 1, \pm 2,\dots$, and to assign to each I_i one of the functions $F_1,\dots,F_k$, which we shall denote by F_{I_i}, so that the function given by $F(x(\cdot),t) = F_{I_i}(x(\cdot),t)$ for $t \in I_i$, $i = \pm 1, \pm 2,\dots$, and $x \in C(I,X)$, satisfies

$$\left|\int_{\tau_1}^{\tau_2} \{\textstyle\sum_{j=1}^{k} p_j(t)F_j(x(\cdot),t) - F(x(\cdot),t)\}dt\right| < \varepsilon$$

for every τ_1,τ_2 in I' and $x \in C(I,X)_{\Psi,K}$.

Proof: The proof follows that given by Gamkrelidze [10; lemma 4.1] where one takes the compact set $C(I,X)_{\Psi,K}$ as the domain of the $F_i(\cdot,t)$. Actually, no use is made of the fact that one function $m \in L^1(I',R)$ bounds all k of the functions F_j; the proof is valid if the bounding function depends on the F_j as in the definition of quasi-convexity.

Let $\mathcal{U} \subset R^r$, $U\colon I' \to$ subsets of $\mathcal{U}$. Define

$$\Omega = \{u\colon u \text{ measurable on } I',\ u(t) \in U(t) \text{ for } t \in I'\}.$$

Consider the controlled system

$$x_{t_0} \in \Phi$$

$$\frac{d}{dt} D(x(\cdot),t) = f(x(\cdot),u(t),t) \quad \text{a.e. on } I'$$

where $\Phi \subset C([\alpha_0,t_0],\mathscr{G})$, $x \in C(I,\mathscr{G})$, $u \in \Omega$.

Lemma 3.3. Assume (2.3)-(2.5), (3.1), f defined on $C(I,\mathscr{G}) \times \mathscr{U} \times I'$, each f^i is C^1 in x, Borel-measurable in (u,t). Also, given compact $X \subset \mathscr{G}$ and $u \in \Omega$, there exists $m \in L^1(I',R)$ such that $|f(x(\cdot),u(t),t)| \leq m(t)$, $\|df[x(\cdot),u(t),t;\cdot]\| \leq m(t)$ for each $t \in I'$, $x \in C(I,X)$; where df is the Frechet differential of f with respect to x. Then the family

$$\mathscr{F} = \{F(x(\cdot),t)\colon F(x(\cdot),t) = f(x(\cdot),u(t),t) \text{ for some } u \in \Omega\}$$

is quasi-convex.

Proof: The assumptions yield a) and b) of the definition immediately. For part c), we use lemma 3.2. Since at each time t, $F_\beta(x(\cdot),t) = F_j(x(\cdot),t)$, j one of $\{1,\dots,k\}$, clearly

$$|F_\beta(x(\cdot),t)| \leq \textstyle\sum_{i=1}^k m_i(t), \quad \|dF_\beta[x(\cdot),t;\cdot]\| \leq \textstyle\sum_{i=1}^k m_i(t).$$

(c') is the conclusion of lemma 3.2. In the proof of lemma 3.2, I' is divided into sub-intervals I'_α in a manner independent of the multipliers $p_i(t)$. Thus the I'_α may be taken as fixed, once given $F_1,\dots,F_k$, and ε. Let the $I'_{\alpha,i}$ (the subset of I'_α on which $F_\beta = F_i$, $\operatorname{meas}(I'_{\alpha,i}) = \int_{I'_\alpha} p_i(t)dt$) be taken in order in I'_α, so that they vary continuously with $\beta = (p_1,\dots,p_k)$. Then by the measurable bounds on F_β and the F_i, and the definition of $G(x(\cdot),t;\beta)$, (c") holds.

We now set up the main theorem to be used in proving necessary conditions. Let $\mathscr{I} = [\alpha_0, t_1] \subset I$, $t_1 > t_0$, $\mathscr{F}$ a quasi-convex family, and $\Phi \subset C([\alpha_0, t_0], \mathscr{G})$. For D satisfying (2.3)-(2.5), define

$$(3.2) \qquad Q(t_1) = \{x \in C(\mathscr{I}, \mathscr{G}): x_{t_0} \in \Phi, \frac{d}{dt} D(x(\cdot), t) = f(x(\cdot), t) \text{ a.e. on } \mathscr{I}' = [t_0, t_1] \text{ for some } f \in \mathscr{F}\}.$$

Assume $z \in C(\mathscr{I}, \mathscr{G})$ satisfies

$$(3.3) \qquad \frac{d}{dt} D(z(\cdot), t) = f^*(z(\cdot), t) \quad \text{a.e. on } \mathscr{I}',$$

f^* a fixed element of $\mathscr{F}$, and

$$(3.4) \qquad z_{t_0} = \varphi^*,$$

φ^* a fixed element of Φ. Clearly, $z \in Q(t_1)$. We denote the elements of $\Phi - \varphi^*$ by $\delta\varphi$, and the elements of $[\mathscr{F}] - f^*$ by δf, where $[\mathscr{F}]$ is the convex hull of $\mathscr{F}$. Let $\mathscr{P}$ be the maximal convex set such that $\varphi^* \in \mathscr{P} \subset \Phi$. $\mathscr{P} - \varphi^*$ is a convex set containing the zero function.

By a) and b) of the definition of quasi-convex families, there exists an $n \times n$-matrix valued function $\eta^*(t, \theta)$ representing the Frechet differential $df^*[z(\cdot), t; \cdot]$, so that (2.7), (2.8) are satisfied. This η^* will be fixed for the rest of the discussion. Let $Y(s,t)$ be the matrix-valued function described by (2.9)-(2.11) for $\eta = \eta^*$. For each $\delta\varphi \in \mathscr{P} - \varphi^*$ and $\delta f \in [\mathscr{F}] - f^*$, define $\delta x(\cdot; \delta\varphi, \delta f) \in C(\mathscr{I}, R^n)$ by

$$(3.5) \qquad \delta x(t;\delta\varphi,\delta f) = Y(t_0,t)D(\delta\varphi,t_0) + \int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^+} d_\alpha[Y(\alpha,t)]\mu(\alpha,\sigma) + \int_{t_0}^{t} Y(\alpha,t)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma) + \int_{t_0}^{t} Y(\alpha,t)\delta f(z(\cdot),\alpha)d\alpha$$

for all $t \in \mathscr{I}'$, and

$$(3.6) \qquad \delta x_{t_0}(\cdot;\delta\varphi,\delta f) = \delta\varphi.$$

Now we define $\mathscr{M} \subset C(\mathscr{I},R^n)$ by

$$(3.7) \qquad \mathscr{M} = \{\delta x(\cdot;\delta\varphi,\delta f),\ \delta\varphi \in \mathscr{P}-\varphi^*,\ \delta f \in [\mathscr{F}]-f^*\}.$$

Since (3.5) is linear in $\delta\varphi$ and δf, and both $\mathscr{P}-\varphi^*$ and $[\mathscr{F}]-f^*$ are convex, if $\delta x_i = \delta x(\cdot;\delta\varphi_i,\delta f_i) \in \mathscr{M}$, $i = 1,\ldots,r$, then for $\beta \in P^r$

$$(3.8) \qquad \begin{aligned}\Sigma_{i=1}^r \beta^i \delta x_i &= \Sigma_{i=1}^r \beta^i \delta x(\cdot;\delta\varphi_i,\delta f_i) \\ &= \delta x(\cdot;\Sigma_{i=1}^r \beta^i \delta\varphi_i,\ \Sigma_{i=1}^r \beta^i \delta f_i)\end{aligned}$$

is in $\mathscr{M}$. Thus $\mathscr{M}$ is convex.

Theorem 3.1. Given the assumptions above, D also satisfies (3.1), and a finite set $A = \{\delta x_1,\ldots,\delta x_r\} \subset \mathscr{M}$; then for every $\nu > 0$ and $\zeta \in (0,1)$, there exists ε', $0 < \varepsilon' \leq \zeta$, and a continuous map $\Theta: P^r \to Q(t_1)$ such that

$$\left\|\frac{\Theta(\beta)-z}{\mathcal{E}'} - \sum_{i=1}^{r}\beta^{i}\delta x_{i}\right\| < \nu \quad \text{for all} \quad \beta = (\beta^{1},\ldots,\beta^{r}) \in P^{r}.$$

(Both Θ and $\mathcal{E}'$ depend on A, ν, and ζ).

Proof: By the definition of $\mathscr{M}$, there exist $\delta f_i \in [\mathscr{F}]-f^*$ and $\delta\varphi_i \in \mathscr{P}-\varphi^*$, $i = 1,\ldots,r$, such that $\delta x_i = \delta x(\cdot;\delta\varphi_i,\delta f_i)$ for each $i = 1,\ldots,r$. Hence there exist $f_1,\ldots,f_q \in \mathscr{F}$, $(\beta_i^1,\ldots,\beta_i^q) \in P^q$, $i = 1,\ldots,r$, such that $\delta f_i = \sum_{j=1}^{q}\beta_i^j f_j - f^*$, $i = 1,\ldots,r$. For each $\beta \in P^r$, let

$$(3.9) \qquad \delta x(\cdot;\beta) = \sum_{i=1}^{r}\beta^{i}\delta x_{i}, \quad \delta\varphi(\cdot;\beta) = \sum_{i=1}^{r}\beta^{i}\delta\varphi_{i}, \text{ and}$$

$$\delta f(\cdot,\cdot;\beta) = \sum_{i=1}^{r}\beta^{i}\delta f_{i}.$$

Then, for all $\beta \in P^r$,

$$(3.10) \qquad f^* + \mathcal{E}\,\delta f(\cdot,\cdot;\beta) = (1-\mathcal{E})f^* + \sum_{j=1}^{q}\left(\sum_{i=1}^{r}\mathcal{E}\beta^{i}\beta_{i}^{j}\right)f_{j}.$$

Since $(1-\mathcal{E}) + \sum_{i=1}^{r}\sum_{j=1}^{q}\mathcal{E}\beta^{i}\beta_{i}^{j} = 1$, we have that $f^* + \mathcal{E}\,\delta f(\cdot,\cdot;\beta) \in [\mathscr{F}]$, for all $\mathcal{E} \in [0,1]$ and $\beta \in P^r$. From (3.5), (3.6), (3.8), (3.9), and theorem 2.4, for each $\beta \in P^r$ and $t \in \mathscr{I}'$,

$$(3.11) \qquad \delta x(t;\beta) = \delta\varphi(t_0;\beta) - \int_{\alpha_0}^{t_0} d_\theta[\mu(t_0,\theta)]\delta\varphi(\theta;\beta)$$
$$+ \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]\delta x(\theta;\beta) + \int_{t_0}^{t} df^*[z(\cdot),s;\delta x(\cdot;\beta)]ds$$
$$+ \int_{t_0}^{t} \delta f(z(\cdot),s;\beta)ds.$$

Let X be a compact, convex, subset of $\mathscr{G}$, so that $z(t)$ is an interior point of X for every $t \in \mathscr{I}$. Such a set exists because the closed convex hull of $\{z(t): t \in \mathscr{I}\}$ is a compact subset of the open convex set $\mathscr{G} \subset R^n$. $\mathscr{F}$ quasi-convex implies that there exist functions $m_j \in L^1(\mathscr{I}',R)$, $j = 0,1,\ldots,q$, such that

$$(3.12) \qquad |f^*(\psi,t)| \leq m_0(t), \quad \|df^*[\psi,t;\cdot]\| \leq m_0(t)$$

$$(3.13) \qquad |f_j(\psi,t)| \leq m_j(t), \quad \|df_j[\psi,t;\cdot]\| \leq m_j(t), \quad j = 1,\ldots,q$$

for all $(\psi,t) \in C(\mathscr{I},X) \times \mathscr{I}'$. From (3.9), (3.12), (3.13),

$$(3.14) \qquad \begin{aligned} |\delta f(\psi,t;\beta)| &= \left|\textstyle\sum_{i=1}^{r} \sum_{j=1}^{q} \beta^i \beta_i^j f_j(\psi,t) - \sum_{i=1}^{r} \beta^i f^*(\psi,t)\right| \\ &\leq \textstyle\sum_{j=1}^{q} \sum_{i=1}^{r} \beta^i \beta_i^j |f_j(\psi,t)| + |f^*(\psi,t)| \\ &\leq \textstyle\sum_{j=0}^{q} m_j(t) \end{aligned}$$

for all $\beta \in P^r$, $(\psi,t) \in C(\mathscr{I},X) \times \mathscr{I}'$. Similarly,

$$(3.15) \qquad \|d(\delta f)[\psi,t;\beta;\cdot]\| \leq \textstyle\sum_{j=0}^{q} m_j(t) \quad \text{for all } \beta \in P^q,\ (\psi,t) \in C(\mathscr{I},X)\times\mathscr{I}'.$$

Now let $\varepsilon \in (0,1)$, $K(t) = \sum_{j=0}^{q} m_j(t)$ for $t \in \mathscr{I}'$, and Ψ = convex hull of $\{\varphi^*,\varphi_1,\ldots,\varphi_r\}$, where $\varphi_i = \delta\varphi_i + \varphi^* \in \mathscr{P}$. By the Arzela-Ascoli theorem, Ψ is compact. From (3.10) and the definition of a quasi-convex family, there exist functions $F(\cdot,\cdot;\beta,\varepsilon) \in \mathscr{F}$, defined for each $\beta \in P^r$, satisfying

(3.16) $$|F(\psi,t;\beta,\varepsilon)| \leq K(t), \quad \|dF[\psi,t;\beta,\varepsilon;\cdot]\| \leq K(t)$$

for all $(\psi,t) \in C(\mathscr{I},X) \times \mathscr{I}'$, $\beta \in P^r$, $\varepsilon \in (0,1)$, such that if

(3.17) $$\delta^2 f(\cdot,\cdot;\beta,\varepsilon) = F(\cdot,\cdot;\beta,\varepsilon) - [f^* + \varepsilon\delta f(\cdot,\cdot;\beta)],$$

then

(3.18) $$\left|\int_\sigma^\tau \delta^2 f(x(\cdot),t;\beta,\varepsilon)dt\right| < \varepsilon^2 \text{ for all } [\sigma,\tau] \subset \mathscr{I}',$$

$$\beta \in P^r,\ x \in C(\mathscr{I},X)_{\Psi,K},$$

and

(3.19) $$\lim_{\substack{\beta' \to \beta \\ \beta' \in P^r}} \delta^2 f(x(\cdot),t;\beta',\varepsilon) = \delta^2 f(x(\cdot),t;\beta,\varepsilon)$$

in measure as a function of t over $\mathscr{I}'$, for each $\beta \in P^r$, $\varepsilon \in (0,1)$, $x \in C(\mathscr{I},X)_{\Psi,K}$. From (3.9), (3.17), (3.19)

$$\lim_{\substack{\beta' \to \beta \\ \beta' \in P^r}} F(x(\cdot),t;\beta',\varepsilon) = F(x(\cdot),t;\beta,\varepsilon)$$

in measure as a function of t over $\mathscr{I}'$, for each $\beta \in P^r$, $\varepsilon \in (0,1)$, $x \in C(\mathscr{I},X)_{\Psi.K}$. Thus, by (3.16) and the Lebesgue Dominated Convergence Theorem,

(3.20) $$\lim_{\substack{\beta' \to \beta \\ \beta' \in P^r}} \left|\int_{t_0}^t [F(x(\cdot),\tau;\beta,\varepsilon) - F(x(\cdot),\tau;\beta',\varepsilon)]d\tau\right| = 0$$

uniformly with respect to $t \in \mathscr{I}'$, for all $\beta \in P^r$, $\varepsilon \in (0,1)$, $x \in C(\mathscr{I},X)_{\Psi,K}$.

We now consider the perturbed equation

$$(3.21) \qquad \frac{d}{dt} D(x(\cdot),t) = F(x(\cdot),t;\beta,\varepsilon)$$

$$= f^*(x(\cdot),t) + \varepsilon\,\delta f(x(\cdot),t;\beta) + \delta^2 f(x(\cdot),t;\beta,\varepsilon)$$

a.e. on $\mathscr{I}'$, with initial conditions

$$(3.22) \qquad x_{t_0} = z_{t_0} + \varepsilon\,\delta\varphi(\cdot;\beta) = \varphi^* + \varepsilon\,\delta\varphi(\cdot;\beta).$$

Let $x(\cdot;\beta,\varepsilon)$ be the solution of (3.21), (3.22); then from (3.3), (3.4), for each $t \in \mathscr{I}'$ where $x(t;\beta,\varepsilon)$ is defined,

$$(3.23) \qquad x(t;\beta,\varepsilon) - z(t) = \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta,\varepsilon) - z(\theta)]$$

$$+ D(\varepsilon\,\delta\varphi(\cdot;\beta),t_0)$$

$$+ \varepsilon\int_{t_0}^{t} \delta f(x(\cdot;\beta,\varepsilon),s;\beta)ds$$

$$+ \int_{t_0}^{t} [f^*(x(\cdot;\beta,\varepsilon),s) - f^*(z(\cdot),s)]ds$$

$$+ \int_{t_0}^{t} \delta^2 f(x(\cdot;\beta,\varepsilon),s;\beta,\varepsilon)ds.$$

We will show there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1]$, all $\beta \in P^r$, $x(t;\beta,\varepsilon)$ is defined on $\mathscr{I}$ and takes values in $\mathscr{G}$. This, together with (3.21) and (3.22), implies that $x(\cdot;\beta,\varepsilon) \in Q(t_1)$ for all $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$. We will also show

$$(3.24) \qquad \lim_{\mathcal{E} \to 0} \left\| \frac{x(\cdot;\beta,\mathcal{E})-z}{\mathcal{E}} - \Sigma_{i=1}^{r} \beta^i \delta x_i \right\|_{t_1} = 0$$

uniformly with respect to $\beta \in P^r$, and

$$(3.25) \qquad \lim_{\substack{\beta' \to \beta \\ \beta' \in P^r}} \| x(\cdot;\beta',\mathcal{E}) - x(\cdot;\beta,\mathcal{E}) \|_{t_1} = 0$$

for all $\beta \in P^r$, $\mathcal{E} \in (0,\mathcal{E}_1]$. Thus, given $\nu > 0$, choosing $\mathcal{E}' \in (0,\mathcal{E}_1] \cap (0,\zeta)$ appropriately small and setting $\Theta(\beta) = x(\cdot;\beta,\mathcal{E}')$, we have that $\Theta(\beta)$ is a continuous map of P^r into $Q(t_1)$ satisfying

$$\left\| \frac{\Theta(\beta)-z}{\mathcal{E}'} - \Sigma_{i=1}^{r} \beta^i \delta x_i \right\| < \nu,$$

as in the theorem statement.

First, we show that there exists $\mathcal{E}_1 > 0$ such that if $0 < \mathcal{E} \leq \mathcal{E}_1$, then $x(t;\beta,\mathcal{E})$ is defined and in $\mathcal{G}$ for all $t \in \mathcal{I}$, $\beta \in P^r$.

Let $Z = \{z(t): t \in \mathcal{I}\}$, so Z is a compact subset of $\mathcal{G}$. Also, by the definition of X, Z is in the interior of $X \subset \mathcal{G}$. Choose $\gamma > 0$ such that, for δ as in (2.5), $0 < 1-\delta(\gamma) < 1$. Let

$$\bar{k} = [\text{greatest integer in } \frac{t_1-t_0}{\gamma}] + 1,$$

$$b = [1-\delta(\gamma)]^{-1},$$

$$a = [1 + 2\delta(t_1-\alpha_0)]b,$$

$$\alpha = \Sigma_{i=1}^{r} \| \delta\varphi_i \|.$$

Choose $c > b$ such that

$$\sup_{t\in[t_0+\gamma,t_1]} a \exp\{(b-c)\int_{t-\gamma}^{t} m_0(s)ds\} < 1.$$

Let

$$\tilde{\rho} = [\bar{k}b + a\alpha + b\int_{t_0}^{t_1}\Sigma_{j=0}^{q} m_j(s)ds]\exp\{c\int_{t_0}^{t_1} m_0(s)ds\},$$

and $\rho_1 > 0$ the distance between Z and the complement of X. If

$$\varepsilon_1 = \min\{1, \rho_1/2\tilde{\rho}\}, \tag{3.26}$$

then, from (3.22) and the above choice of constants, for $t \in [\alpha_0, t_0]$, $\beta \in P^r$, and $\varepsilon \in (0, \varepsilon_1]$,

$$|x(t;\beta,\varepsilon) - z(t)| < \rho_1/2.$$

Thus $x(t;\beta,\varepsilon)$ is defined and belongs to X for every $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$, and $t \in [\alpha_0, t_0]$. Assume that, for some $\bar{\beta} \in P^r$, $\bar{\varepsilon} \in (0,\varepsilon_1]$, the solution of (3.21), (3.22) either fails to exist on all of $\mathscr{I}$, or else leaves X. Let

$$\tau = \sup\{t' \in \mathscr{I}': x(t;\bar{\beta},\bar{\varepsilon}) \text{ defined and in } X \text{ for all } t \in [\alpha_0, t']\}.$$

Denote $[\alpha_0, \tau]$ by $\mathscr{I}_0$, $[t_0, \tau]$ by $\mathscr{I}'_0$. Thus $x(t;\bar{\beta},\bar{\varepsilon}) \in X$ for all $t \in \mathscr{I}_0$, and from (3.16), (3.21)

$$\left|\frac{d}{dt} D(x(\cdot;\bar{\beta},\bar{\varepsilon}),t)\right| \leq K(t) \quad \text{a.e. on } \mathscr{I}'_0.$$

By (3.18) and the definition of Ψ, for all $t \in \mathscr{I}'_0$,

$$\left|\int_{t_0}^{t} \delta^2 f(x(\cdot;\bar{\beta},\bar{\varepsilon}),s;\bar{\beta},\bar{\varepsilon})ds\right| < \bar{\varepsilon}^2.$$

From (3.12), (3.14), and (3.23), for all $t \in \mathscr{I}_0'$,

$$\begin{aligned}(3.27)\qquad |x(t;\bar{\beta},\bar{\varepsilon})-z(t)| &\le \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\bar{\beta},\bar{\varepsilon})-z(\theta)]\right| \\ &\quad + |D(\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta}),t_0)| + \bar{\varepsilon}\int_{t_0}^{t} \textstyle\sum_{j=0}^{q} m_j(s)ds \\ &\quad + \int_{t_0}^{t} m_0(s)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_s ds + \bar{\varepsilon}^2 \\ &\le \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\bar{\beta},\bar{\varepsilon})-z(\theta)]\right| \\ &\quad + [1 + \int_{\alpha_0}^{t_0}|d_\theta\mu(t,\theta)|]\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta})\| \\ &\quad + \bar{\varepsilon}\int_{t_0}^{t} \textstyle\sum_{j=0}^{q} m_j(s)ds + \bar{\varepsilon}^2 \\ &\quad + \int_{t_0}^{t} m_0(s)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_s ds.\end{aligned}$$

By induction we will show that, for every $t \in \mathscr{I}_0'$,

$$(3.28)\qquad \|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \le \bar{\varepsilon}\,\tilde{\rho}.$$

For $t \in [t_0,t_0+\gamma]$, by (3.22)

$$\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\bar{\beta},\bar{\varepsilon})-z(\theta)] =$$

$$\begin{aligned} &= \int_{\alpha_0}^{t_0} d_\theta[\mu(t,\theta)][\bar{\varepsilon}\,\delta\varphi(\theta;\bar{\beta})] \\ &\quad + \int_{t_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\bar{\beta},\bar{\varepsilon})-z(\theta)], \end{aligned}$$

and so

$$\begin{aligned} \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\bar{\beta},\bar{\varepsilon})-z(\theta)]\right| &\le \delta(t_1-\alpha_0)\|\bar{\varepsilon}\,\delta\varphi(\cdot;\bar{\beta})\| \\ &\quad + \delta(\gamma)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t . \end{aligned}$$

Hence, from (3.27), for $t \in [t_0,t_0+\gamma]$,

$$\begin{aligned} |x(t;\bar{\beta},\bar{\varepsilon})-z(t)| &\le \delta(t_1-\alpha_0)\|\bar{\varepsilon}\,\delta\varphi(\cdot;\bar{\beta})\| + \delta(\gamma)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \\ &\quad + [1+\delta(t_1-\alpha_0)]\|\bar{\varepsilon}\,\delta\varphi(\cdot;\bar{\beta})\| + \bar{\varepsilon}^2 \\ &\quad + \bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)\,ds + \int_{t_0}^{t} m_0(s)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_s\,ds . \end{aligned}$$

Since $[1+2\delta(t_1-\alpha_0)] > 1$, $x(t;\bar{\beta},\bar{\varepsilon})-z(t) = \bar{\varepsilon}\,\delta\varphi(t;\bar{\beta})$ for $t \in [\alpha_0,t_0]$, and the right-hand side is non-decreasing in t, this may be rewritten

$$\begin{aligned} \|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t &\le [1+2\delta(t_1-\alpha_0)]\|\bar{\varepsilon}\,\delta\varphi(\cdot;\bar{\beta})\| + \bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)\,ds \\ &\quad + \bar{\varepsilon}^2 + \int_{t_0}^{t} m_0(s)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_s\,ds \\ &\quad + \delta(\gamma)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t, \quad t \in [t_0,t_0+\gamma]. \end{aligned}$$

Subtracting $\delta(\gamma)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t$ from both sides and using the definitions of the constants,

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \le b\bar{\varepsilon}^2 + a\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\varepsilon})\| + b\bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)ds$$

$$+ b\int_{t_0}^{t} m_0(s)\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_s ds, \quad t \in [t_0,t_0+\gamma].$$

Gronwall's inequality states that if $y(t) \le \lambda(t) + \int_a^t \mu(s)y(s)ds$, $\lambda \ge 0$, $\mu \ge 0$, then $y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s)\exp\{\int_s^t \mu(\xi)d\xi\}ds$. If λ is absolutely continuous and non-decreasing, we may integrate by parts, obtaining

$$y(t) \le \lambda(t) - \lambda(s)\exp\{\int_s^t \mu(\xi)d\xi\}\Big|_a^t$$

$$+ \int_a^t \dot{\lambda}(s)\exp\{\int_s^t \mu(\xi)d\xi\}ds.$$

Since $\mu \ge 0$ and $\dot{\lambda} \ge 0$ almost everywhere,

$$y(t) \le \lambda(a)\exp\{\int_a^t \mu(\xi)d\xi\} + \exp\{\int_a^t \mu(\xi)d\xi\}\int_a^t \dot{\lambda}(s)ds,$$

so $y(t) \le \lambda(t)\exp\{\int_a^t \mu(\xi)d\xi\}$.

Using this modified Gronwall's inequality,

$$\text{(3.29)} \qquad \|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \le [b\bar{\varepsilon}^2 + a\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta})\|$$

$$+ b\bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)ds]\exp\{b\int_{t_0}^{t} m_0(s)ds\}.$$

Now we make the induction assumption that, for $t \in [t_0, t_0+k\gamma] \subset \mathscr{I}'_0$, $k \geq 1$ an integer,

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \leq [kb\bar{\varepsilon}^2 + a\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta})\|$$

$$+ b\bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)ds]\exp\{c\int_{t_0}^{t} m_0(s)ds\}.$$

Let $t \in [t_0+k\gamma, t_0+(k+1)\gamma]$. At first we assume a solution $x(s;\bar{\beta},\bar{\varepsilon})-z(s)$ with initial function $x_{t-\gamma}(\cdot;\bar{\beta},\bar{\varepsilon})-z_{t-\gamma}$ at initial time $t-\gamma$. Reasoning as before, a form such as (3.29) clearly applies, so

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \leq [b\bar{\varepsilon}^2 + a\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_{t-\gamma}$$

$$+ b\bar{\varepsilon}\int_{t-\gamma}^{t} \Sigma_{j=0}^{q} m_j(s)ds]\exp\{b\int_{t-\gamma}^{t} m_0(s)ds\}.$$

Using the induction assumption,

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \leq [b\bar{\varepsilon}^2 + a\{kb\bar{\varepsilon}^2 + a\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta})\|$$

$$+ b\bar{\varepsilon}\int_{t_0}^{t-\gamma} \Sigma_{j=0}^{q} m_j(s)ds\}\exp\{c\int_{t_0}^{t-\gamma} m_0(s)ds\}$$

$$+ b\bar{\varepsilon}\int_{t-\gamma}^{t} \Sigma_{j=0}^{q} m_j(s)ds]\exp\{b\int_{t-\gamma}^{t} m_0(s)ds\}.$$

By the choice of c, for $t \in [t_0+k\gamma,\ t_0 + (k+1)\gamma]$,

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \leq [(k+1)b\bar{\varepsilon}^2 + a\|\bar{\varepsilon}\delta\varphi(\cdot;\bar{\beta})\|$$

$$+ b\bar{\varepsilon}\int_{t_0}^{t} \Sigma_{j=0}^{q} m_j(s)ds]\exp\{c\int_{t_0}^{t} m_0(s)ds\}.$$

Thus, by induction and the choice of $\bar{k}$, α, and $\tilde{\rho}$, for $t \in \mathcal{I}'_0$,

$$\|x(\cdot;\bar{\beta},\bar{\varepsilon})-z\|_t \leq \bar{\varepsilon}[\bar{k}b + a\alpha + b \int_{t_0}^{t_1} \sum_{j=0}^{q} m_j(s)ds]\exp\{c \int_{t_0}^{t_1} m_0(s)ds\}$$

$$= \bar{\varepsilon}\,\tilde{\rho},$$

which is (3.28). From (3.26), $|x(t;\bar{\beta},\bar{\varepsilon})-z(t)| \leq \rho_1/2$ for all $t \in \mathcal{I}_0$; in particular $x(\tau;\bar{\beta},\bar{\varepsilon}) \in$ interior X. Combining (3.16), (3.21), (3.22), $x(\cdot;\bar{\beta},\bar{\varepsilon}) \in C(\mathcal{I}_0,X)_{\Psi,K}$, which by lemma 3.1 is compact. Hence theorem 2.3 implies $x(\cdot;\bar{\beta},\bar{\varepsilon})$ can be continued beyond τ. Since $x(\tau;\bar{\beta},\bar{\varepsilon}) \in$ interior X, this contradicts the definition of τ.

We conclude that there exists $\varepsilon_1 \in (0,1]$, given by (3.26), such that $x(t;\beta,\varepsilon)$ is defined and takes values in X for all $t \in \mathcal{I}$, whenever $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$. By the same arguments, with $\bar{\beta}$ replaced by arbitrary $\beta \in P^r$, $\bar{\varepsilon}$ by arbitrary $\varepsilon \in (0,\varepsilon_1]$, we conclude that for all $t \in \mathcal{I}$, $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$,

(3.30) $$|x(t;\beta,\varepsilon)-z(t)| < \varepsilon\,\tilde{\rho} \leq \rho_1,$$

and so

(3.31) $$x(\cdot;\beta,\varepsilon) \in C(\mathcal{I},X)_{\Psi,K}.$$

We now prove (3.24). From (3.23), whenever $t \in \mathcal{I}$, $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$,

$$(3.32)\qquad \frac{x(t;\beta,\varepsilon)-z(t)}{\varepsilon} = \frac{1}{\varepsilon}\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta,\varepsilon)-z(\theta)]$$

$$+ D(\delta\varphi(\cdot;\beta),t_0) + \int_{t_0}^{t} \delta f(x(\cdot;\beta,\varepsilon),s;\beta)ds$$

$$+ \frac{1}{\varepsilon}\int_{t_0}^{t} [f^*(x(\cdot;\beta,\varepsilon), s) - f^*(z(\cdot),s)]ds$$

$$+ \frac{1}{\varepsilon}\int_{t_0}^{t} \delta^2 f(x(\cdot;\beta,\varepsilon),s;\beta,\varepsilon)ds$$

$$= \frac{1}{\varepsilon}\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta,\varepsilon)-z(\theta)] + D(\delta\varphi(\cdot;\beta),t_0)$$

$$+ \int_{t_0}^{t} \delta f(z(\cdot),s;\beta)ds + \lambda(t;\beta,\varepsilon)$$

$$+ \frac{1}{\varepsilon}\int_{t_0}^{t} df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]ds,$$

where

$$\lambda(t;\beta,\varepsilon) = \int_{t_0}^{t} [\delta f(x(\cdot;\beta,\varepsilon),s;\beta) - \delta f(z(\cdot),s;\beta)]ds$$

$$+ \frac{1}{\varepsilon}\int_{t_0}^{t} \{f^*(x(\cdot;\beta,\varepsilon),s) - f^*(z(\cdot),s)$$

$$- df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]\}ds$$

$$+ \frac{1}{\varepsilon}\int_{t_0}^{t} \delta^2 f(x(\cdot;\beta,\varepsilon),s;\beta,\varepsilon)ds.$$

Using (3.15), (3.18), (3.30), and (3.31),

$$|\lambda(t;\beta,\varepsilon)| \leq \int_{t_0}^{t_1} \Sigma_{j=0}^{q} m_j(s) \|x(\cdot;\beta,\varepsilon)-z\|_s ds$$

$$+ \int_{t_0}^{t_1} \frac{1}{\varepsilon} \, | f^*(x(\cdot;\beta,\varepsilon),s) - f^*(z(\cdot),s)$$

$$- df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]| ds + \varepsilon$$

$$< \varepsilon[1 + \tilde{\rho} \int_{t_0}^{t_1} \Sigma_{j=0}^{q} m_j(s) ds]$$

$$+ \int_{t_0}^{t_1} \frac{1}{\varepsilon} \, | f^*(x(\cdot;\beta,\varepsilon),s) - f^*(z(\cdot),s)$$

$$- df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]| ds.$$

We note that the entire segment in $C(\mathscr{I},\mathscr{G})$ joining $x_t(\cdot;\beta,\varepsilon)$ and z_t lies in $C([\alpha_0,t],X)$ for all $t \in \mathscr{I}$, $\beta \in P^r$, $\varepsilon \in (0,\varepsilon_1]$. Also, $f^* \in \mathscr{F}$, and so is C^1 in x. Thus, by (3.30) and the definition of the Frechet derivative,

$$\lim_{\varepsilon \to 0^+} |\tfrac{1}{\varepsilon}\{f^*(x(\cdot;\beta,\varepsilon),s)-f^*(z(\cdot),s)-df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]\}|$$

$$\leq \lim_{\varepsilon \to 0^+} \tilde{\rho}\cdot \frac{1}{\|x(\cdot;\beta,\varepsilon)-z\|_s} \, | f^*(x(\cdot;\beta,\varepsilon),s)-f^*(z(\cdot),s)$$

$$-df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]|$$

$$= 0 \quad \text{uniformly with respect to} \quad \beta \in P^r \quad \text{for all} \quad s \in \mathscr{I}'.$$

From (3.12) and (3.30),

$$\frac{1}{\varepsilon}| f^*(x(\cdot;\beta,\varepsilon),s) - f^*(z(\cdot),s) - df^*[z(\cdot),s;x(\cdot;\beta,\varepsilon)-z(\cdot)]|$$

$$\leq \frac{1}{\varepsilon}\{ \sup_{\xi\in[0,1]} \|df^*[\xi x(\cdot;\beta,\varepsilon) + (1-\xi)z(\cdot),s;\cdot]\|$$

$$+ \|df^*[z(\cdot),s;\cdot]\|\}\|x(\cdot;\beta,\varepsilon)-z\|$$

$$\leq \frac{1}{\varepsilon}\{2m_0(s)\}\varepsilon\tilde{\rho} = 2\tilde{\rho}m_0(s).$$

By these estimates and the Lebesgue Dominated Convergence theorem,

$$\lim_{\varepsilon \to 0^+} \lambda(t;\beta,\varepsilon) = 0 \quad \text{uniformly with respect to} \quad t \in \mathscr{I}', \ \beta \in P^r.$$

Let

$$\hat{\lambda}(\varepsilon) = \max_{(t,\beta) \in \mathscr{I}' \times P^r} |\lambda(t;\beta,\varepsilon)|;$$

note that $\lim_{\varepsilon \to 0^+} \hat{\lambda}(\varepsilon) = 0$. If

$$q(t;\beta,\varepsilon) = \frac{x(t;\beta,\varepsilon) - z(t)}{\varepsilon} - \sum_{i=1}^r \beta^i \delta x_i(t),$$

then by (3.9), (3.11), and (3.32)

$$q(t;\beta,\varepsilon) = \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)q(\theta;\beta,\varepsilon) + \lambda(t;\beta,\varepsilon)$$

$$+ \int_{t_0}^{t} df^*[z(\cdot),s;q(\cdot;\beta,\varepsilon)]ds.$$

By (3.9) and (3.22), $q_{t_0}(\cdot;\beta,\varepsilon) = 0$. Again choosing $\gamma > 0$ so that $0 < 1-\delta(\gamma) < 1$, and setting $b = [1-\delta(\gamma)]^{-1}$, and using (3.12), for

$t \in [t_0, t_0+\gamma]$,

$$|q(t;\beta,\varepsilon)| \leq \delta(\gamma)\|q(\cdot;\beta,\varepsilon)\|_t + \hat{\lambda}(\varepsilon) + \int_{t_0}^{t} m_0(s)\|q(\cdot;\beta,\varepsilon)\|_s ds.$$

Proceeding as in the proof of (3.28), we combine terms and apply Gronwall's inequality to obtain

$$\|q(\cdot;\beta,\varepsilon)\|_t \leq b\hat{\lambda}(\varepsilon)\exp\{b\int_{t_0}^{t} m_0(s)ds\}, \quad t \in [t_0, t_0+\gamma].$$

For an initial time $\sigma > t_0$ and $t \in [\sigma, \sigma+\gamma]$,

$$\|q(\cdot;\beta,\varepsilon)\|_t \leq [1 + \delta(t_1-\alpha_0)]\|q(\cdot;\beta,\varepsilon)\|_\sigma + \delta(\gamma)\|q(\cdot;\beta,\varepsilon)\|_t$$

$$+ \hat{\lambda}(\varepsilon) + \int_{\sigma}^{t} m_0(s)\|q(\cdot;\beta,\varepsilon)\|_s ds.$$

$$\|q(\cdot;\beta,\varepsilon)\|_t \leq [b\{1+\delta(t_1-\alpha_0)\}\|q(\cdot;\beta,\varepsilon)\|_\sigma + b\hat{\lambda}(\varepsilon)]\exp\{b\int_{\sigma}^{t} m_0(s)ds\}.$$

Choosing $\bar{k}$ as before and $c > b$ so that

$$\sup_{t\in[t_0+\gamma,t_1]} b[1+\delta(t_1-\alpha_0)]\exp\{(b-c)\int_{t-\gamma}^{t} m_0(s)ds\} < 1,$$

the form after the induction step is that for all $t \in [t_0, t_1]$, $\beta \in P^r$, $\varepsilon \in (0, \varepsilon_1]$,

$$\|q(\cdot;\beta,\varepsilon)\|_t \leq \bar{k}b\hat{\lambda}(\varepsilon)\exp\{c\int_{t_0}^{t_1} m_0(s)ds\}.$$

Thus $\lim_{\varepsilon\to 0^+}\|q(\cdot;\beta,\varepsilon)\|_{t_1} = 0$ uniformly with respect to $\beta \in P^r$, which is (3.24).

We complete the theorem proof by showing that (3.25) holds; since $\varepsilon \in (0,\varepsilon_1]$ is fixed in the remainder of the proof, we drop it as an argument in $x(t;\beta,\varepsilon)$ and $F(x(\cdot),t;\beta,\varepsilon)$. Fix $\beta \in P^r$. It follows from (3.21), (3.22) that for all $\beta' \in P^r$, $t \in \mathscr{I}'$,

$$(3.33) \qquad \begin{aligned} |x(t;\beta)-x(t;\beta')| &\le \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta)-x(\theta,\beta')]\right| \\ &\quad + \left|D\left(\textstyle\sum_{i=1}^{r}(\beta^i-\beta'^i)\delta\varphi_i,t_0\right)\right| \\ &\quad + \left|\int_{t_0}^{t}[F(x(\cdot;\beta),s;\beta)-F(x(\cdot;\beta'),s;\beta')]ds\right| \\ &\le \Lambda(t;\beta') + \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta)-x(\theta,\beta')]\right| \\ &\quad + \int_{t_0}^{t}|F(x(\cdot;\beta),s;\beta') - F(x(\cdot;\beta'),s;\beta')|ds \end{aligned}$$

where, for $\alpha = \sum_{i=1}^{r}|\delta\varphi_i|$,

$$\begin{aligned} \Lambda(t;\beta') &= \alpha \textstyle\sum_{i=1}^{r}|\beta^i-\beta'^i|[1+\delta(t_1-\alpha_0)] \\ &\quad + \left|\int_{t_0}^{t}[F(x(\cdot;\beta),s;\beta) - F(x(\cdot;\beta),s;\beta')]ds\right| . \end{aligned}$$

It follows from (3.20) and (3.31) that, if we set $\hat{\Lambda}(\beta') = \max_{t\in\mathscr{I}'} \Lambda(t;\beta')$ for all $\beta' \in P^r$, then $\lim_{\substack{\beta' \to \beta \\ \beta' \in P^r}} \hat{\Lambda}(\beta') = 0$. By (3.16) and (3.33), for $t \in \mathscr{I}'$,

$$\begin{aligned} |x(t;\beta)-x(t;\beta')| &\le \hat{\Lambda}(\beta') + \left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta)-x(\theta;\beta')]\right| \\ &\quad + \int_{t_0}^{t} K(t)\|x(\cdot;\beta)-x(\cdot;\beta')\|_s ds. \end{aligned}$$

Choosing γ and b as before, for $t \in [t_0, t_0+\gamma]$

$$\left|\int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)][x(\theta;\beta)-x(\theta;\beta')]\right| \leq \delta(t_1-\alpha_0)\|\delta\varphi(\cdot;\beta)-\delta\varphi(\cdot;\beta')\|$$

$$+ \delta(\gamma)\|x(\cdot;\beta)-x(\cdot;\beta')\|_t .$$

$$\|x(\cdot;\beta)-x(\cdot;\beta')\|_t \leq 2b\hat{\Lambda}(\beta') + b\int_{t_0}^{t} K(s)\|x(\cdot;\beta)-x(\cdot;\beta')\|_s ds.$$

$$\|x(\cdot;\beta)-x(\cdot;\beta')\|_t \leq 2b\hat{\Lambda}(\beta')\exp\{b\int_{t_0}^{t} K(s)ds\}.$$

Choosing $\bar{k}$ as before, and $c > b$ so that

$$\sup_{t\in[t_0+\gamma,t_1]} b\delta(t_1-\alpha_0)\exp\{(b-c)\int_{t-\gamma}^{t} K(s)ds\} < 1,$$

we obtain by induction that, for $t \in \mathscr{I}'$,

$$\|x(\cdot;\beta)-x(\cdot;\beta')\|_t \leq (\bar{k}+1)\hat{\Lambda}(\beta')\exp\{c\int_{t_0}^{t_1} K(s)ds\}.$$

Thus $\lim\limits_{\substack{\beta' \to \beta \\ \beta \to P^r}} \|x(\cdot;\beta)-x(\cdot;\beta')\|_{t_1} = 0$, which is (3.25), and the proof is complete.

4. Necessary Conditions

We first formulate a control problem with terminal manifold in R^n. Let $L_{-\mu},\ldots,L_0,\ldots,L_m$ be given real valued, C^1 functions defined on $C([\alpha_0,t_0],\mathscr{G}) \times \mathscr{G} \times [t_0,a)$, $\mathscr{F}$ be a quasi-convex family, $\Phi \subset C([\alpha_0,t_0],\mathscr{G})$, and $D(x(\cdot),t)$ satisfy (2.3)-(2.5) and (3.1). Define

$$\mathscr{E} = \{(x,t): t \in [t_0,a),\ x \in Q(t)\}$$

where $Q(t)$ is given by (3.2). Let the functions $\varphi_i: \mathscr{E} \to R$, $i = -\mu,\ldots,m$, be given by $\varphi_i(x,t_1) = L_i(x_{t_0},x(t_1),t_1)$.

Problem 4.1: We wish to find $(z,t^*) \in \mathscr{E}$ such that

a) $\varphi_i(z,t^*) \leq 0$ for $i = -\mu,\ldots,-1$,

b) $\varphi_i(z,t^*) = 0$ for $i = 1,\ldots,m$,

c) $\varphi_0(z,t^*) \leq \varphi_0(x,t_1)$ for all $(x,t_1) \in \mathscr{E}$ which satisfy a) and b).

For such a (z,t^*), let φ^*, f^*, and Y be given by (3.3), (3.4), and the remarks after (3.4).

Theorem 4.1. Given the assumptions above, let (z,t^*) be a solution of problem 4.1 such that $\dot{z}(t^*)$ exists and either $\Phi = \{\varphi^*\}$ or $\varphi^* \in$ interior Φ. Then there exists a row n-vector valued function $\bar{\psi}$ defined on $[t_0,\infty)$, and real numbers α^i, $i = -\mu,\ldots,m$, such that

i) $\alpha^i \leq 0$ for $i \leq 0$, $\alpha^i = 0$ for all $i \in \{-1,\ldots,-\mu\}$ such that $\varphi_i(z,t^*) < 0$, $\sum_{i=-\mu}^m |\alpha^i| > 0$.

ii) $\bar{\psi}$ is given by

$$\bar{\psi}(s) = \Sigma_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z(t^*), t^*; 0, Y(s,t^*), 0] \tag{4.1}$$

where dL_i is the Frechet differential of L_i, and satisfies

$$\bar{\psi}(s) = 0, \quad s > t^*, \tag{4.2}$$

$$\bar{\psi}(t^*) = \Sigma_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z(t^*), t^*; 0, E, 0], \tag{4.3}$$

$$\bar{\psi}(t^*)\dot{z}(t^*) = -\Sigma_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z(t^*), t^*; 0, 0, 1], \tag{4.4}$$

$$\bar{\psi}(s) = \bar{\psi}(t^*) + \int_s^{t^{*+}} d_\alpha[\bar{\psi}(\alpha)]\mu(\alpha,s) - \int_s^{t^*} \bar{\psi}(\alpha)\eta^*(\alpha,s)d\alpha \tag{4.5}$$

for $s \in [t_0, t^*)$,

$$\bar{\psi}(t_0) = -\Sigma_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z(t^*), t^*; Y_{t_0}(t_0,\cdot), 0, 0] \tag{4.6}$$

(note: (4.6) need not hold if $\Phi = \{\varphi^*\}$). If, in addition, $\Sigma_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z(t^*), t^*; \cdot, \cdot, 0] = 0$ and $\alpha^i \leq 0$ for $i \in \{-\mu, \ldots, 0\}$ imply $\alpha^i = 0$, $i = -\mu, \ldots, m$, then $\bar{\psi}$ is non-zero on a subset of $[t_0, t^*]$ of positive measure.

iii) the following maximum condition holds:

$$\int_{t_0}^{t^*} \bar{\psi}(s)f(z(\cdot),s)ds \leq \int_{t_0}^{t^*} \bar{\psi}(s)f^*(z(\cdot),s)ds \quad \text{for all} \quad f \in \mathscr{F}.$$

Proof: The definitions and conditions referred to in the following are contained in [22]. Since (z, t^*) is a solution, it is a (φ, Φ, Z) extremal, where

$$\varphi = (\varphi_1,\dots,\varphi_m)$$

$$\phi = (\varphi_0-\varphi_0(z,t^*),\varphi_{-1},\dots,\varphi_{-\mu}),$$

$$\mathscr{Z} = R^{\mu+1},$$

$$Z = \{\gamma \in R^{\mu+1}\colon \gamma_j < 0,\ j = 1,\dots,\mu+1\}.$$

Choose $\varepsilon \in (0,a-t^*)$ such that $z \in Q(t^*+\varepsilon)$, (such an ε exists by theorem 2.3) and set $t' = t^*+\varepsilon$. Define $\mathscr{L}_1 = Q(t') \times [t_0,t']$. Let $\mathscr{Y} = C([\alpha_0,t'],R^n) \times R$. Clearly condition 6.1 is satisfied. By theorem 3.1 above, condition 6.2 is satisfied by $M = \mathscr{M} \times [-\varepsilon,\varepsilon]$ and $\Theta(\beta) = (\theta(\beta),t^* + \varepsilon' \sum_{i=1}^r \beta^i\tau_i)$, where $\mathscr{M}$ is given by (3.7) and $\theta(\beta)$ is the map $\Theta(\beta)$ of theorem 3.1. Theorem 6.4 in [22] implies conditions 6.3 and 6.4 are satisfied, with

$$h(y,\tau) = (dL_1[z_{t_0},z(t^*),t^*;y_{t_0},y(t^*) + \dot z(t^*)\tau,\tau],\dots,$$
$$dL_m[z_{t_0},z(t^*),t^*;y_{t_0},y(t^*) + \dot z(t^*)\tau,\tau]), \tag{4.7}$$

$$\hat h(y,\tau) = (dL_0[z_{t_0},z(t^*),t^*;y_{t_0},y(t^*) + \dot z(t^*)\tau,\tau],\dots,$$
$$dL_{-\mu}[z_{t_0},z(t^*),t^*;y_{t_0},y(t^*) + \dot z(t^*)\tau,\tau]). \tag{4.8}$$

By theorems 6.1 and 5.1 of [22], there exist vectors $\alpha \in R^m$ and $\hat\alpha = (\alpha^0,\alpha^{-1},\dots,\alpha^{-\mu}) \in R^{\mu+1}$ such that

$$\alpha\cdot h(y,\tau) + \hat\alpha\cdot\hat h(t,\tau) \le 0 \quad \text{for all } (y,\tau) \in M, \tag{4.9}$$

$$|\alpha| + |\hat\alpha| > 0, \tag{4.10}$$

(4.11) $$\alpha^i \leq 0 \quad \text{for} \quad i = 0,-1,\ldots,-\mu,$$

(4.12) $$\hat{\alpha}\cdot\Phi(z,t^*) = 0.$$

From (4.11), (4.12) we obtain that $\alpha^i = 0$ for all $i \in \{-1,\ldots,-\mu\}$ such that $\varphi_i(z,t^*) < 0$; this and (4.10), (4.11) are part i). Substituting (4.7) and (4.8) in (4.9), for all $(y,\tau) \in M$,

$$\Sigma_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0},z(t^*),t^*;y_{t_0},y(t^*) + \dot{z}(t^*)\tau,\tau] \leq 0.$$

From (3.7) and the definition of M, for all $\delta\varphi \in \mathscr{P}-\varphi^*$, $\delta f \in [\mathscr{F}]-f^*$, and $\tau \in [-\mathcal{E},\mathcal{E}]$,

(4.13) $$\Sigma_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0},z(t^*),t^*;\delta\varphi,\delta x(t^*;\delta\varphi,\delta f) + \dot{z}(t^*)\tau,\tau] \leq 0.$$

Let $\delta\varphi = 0$, $\tau = 0$; from (3.5) and (4.13), for all $\delta f \in [\mathscr{F}]-f^*$,

$$\Sigma_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0},z(t^*),t^*;0,\int_{t_0}^{t^*} Y(s,t^*)\delta f(z(\cdot),s)ds,0] \leq 0.$$

Using (4.1) to define $\overline{\psi}(s)$, interchanging the dL_i and the integral, and noting that $\delta f = f-f^*$ for some $f \in [\mathscr{F}]$, this becomes iii). (2.11) and (4.1) imply (4.2), (2.10) and (4.1) imply (4.3).

Let $\delta\varphi = 0$, $\delta f = 0$, $\tau \in [-\mathcal{E},\mathcal{E}]$. From (4.13) and (4.3),

$$\overline{\psi}(t^*)\dot{z}(t^*)\tau \leq -\Sigma_{i=-\mu}^{m}dL_i[z_{t_0},z(t^*),t^*;0,0,\tau].$$

Since this is linear and τ is symmetric about zero, we have (4.4). Applying

(4.1) to (2.9), then interchanging the dL_i and the integral, we obtain (4.5).

Now let $\delta f = 0$, $\tau = 0$. If $\varphi^* \in$ interior Φ, there exists $\rho > 0$ such that if

$$\mathscr{P}' = \{\varphi \in C([\alpha_0,t_0],R^n)\colon \|\varphi-\varphi^*\| < \rho\}, \text{ then } \mathscr{P}' \subset \mathscr{P}.$$

From (3.5), (4.13), and the fact that $-\delta\varphi \in \mathscr{P}'-\varphi^*$ if $\delta\varphi \in \mathscr{P}'-\varphi^*$; for all $\delta\varphi \in \mathscr{P}'-\varphi^*$,

$$\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0},z(t^*),t^*;\delta\varphi,\cdot,0][Y(t_0,t^*)D(\delta\varphi,t_0)$$
$$+\int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^{*+}} d_\alpha[Y(\alpha,t^*)]\mu(\alpha,\sigma) + \int_{t_0}^{t^*} Y(\alpha,t^*)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma)] = 0.$$

Using (4.1) and interchanging the dL_i and the integrals, we obtain for all $\delta\varphi \in \mathscr{P}'-\varphi^*$,

$$\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0},z(t^*),t^*;\delta\varphi,0,0] + \bar{\psi}(t_0)D(\delta\varphi,t_0)$$
$$+\int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^{*+}} d_\alpha[\bar{\psi}(\alpha)]\mu(\alpha,\sigma) + \int_{t_0}^{t^*} \bar{\psi}(\alpha)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma) = 0.$$

Let $P = \{\xi \in R^n\colon |\xi| < \rho\}$. For each $\xi \in P$, each $\zeta \in (\alpha_0,t_0)$, define $\delta\tilde{\varphi}_{\xi,\zeta} \in \mathscr{P}'-\varphi^*$ by

$$(4.14) \qquad \delta\tilde{\varphi}_{\xi,\zeta}(s) = \begin{cases} 0 \;, & \alpha_0 \le s \le \zeta \\ \dfrac{(s-\zeta)}{(t_0-\zeta)}\xi, & \zeta \le s \le t_0 \end{cases}$$

Note that, for all such ζ, $\delta\tilde{\varphi}_{\xi,\zeta}(t_0) = \xi$. Thus, setting $\delta\varphi = \delta\tilde{\varphi}_{\xi,\zeta}$ in the last equation, and passing to the limit as $\zeta \to t_0^-$,

$$\{\textstyle\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0}, z(t^*), t^*; Y_{t_0}(t_0,\cdot), 0, 0] + \bar{\psi}(t_0)\}\xi = 0$$

for all $\xi \in P$. Thus the quantity in braces vanishes, which is (4.6).

There remains only to show that, given the additional assumption, $\bar{\psi}$ is non-zero on a subset of $[t_0,t^*]$ of positive measure. For every $x \in C([\alpha_0,t^*],R^n)$ with $\|x_{t_0}\| < \rho$, define the functions w_x on $[t_0,t^*]$ and φ_x on $[\alpha_0,t_0]$ by

$$w_x(t) = \frac{d}{dt}[D(x(\cdot),t)] - df^*[z(\cdot),t;x(\cdot)] \quad \text{a.e. on } [t_0,t^*],$$

$$\varphi_x = x_{t_0}.$$

Then, by theorem 2.4, x is given by

$$x(t) = Y(t_0,t)D(\varphi_x,t_0) + \int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^+} d_\alpha[Y(\alpha,t)]\mu(\alpha,\sigma) + \int_{t_0}^{t} Y(\alpha,t)\eta^*(\alpha,\sigma)d\alpha\}\varphi_x(\sigma) + \int_{t_0}^{t} Y(\alpha,t)w_x(\alpha)d\alpha. \tag{4.15}$$

Define the functional on $C([\alpha_0,t^*],R^n) \times R$

$$\tilde{L}(\cdot,\cdot) = \textstyle\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0}, z(t^*), t^*; \cdot, \cdot].$$

From (4.13) and the fact that $\delta\varphi \in \mathscr{P}' - \varphi^*$ implies $-\delta\varphi \in \mathscr{P}' - \varphi^*$, we obtain that, for all $\delta\varphi \in \mathscr{P}' - \varphi^*$, $\tilde{L}(\delta x(\cdot;\delta\varphi,0),0) = 0$. Thus, from (4.1) and (4.15)

$\tilde{L}(x,0) = \int_{t_0}^{t*} \overline{\Psi}(s)w_x(s)ds$. By the additional assumption and (4.10), $\tilde{L}(\cdot,0) \neq 0$, and so $\overline{\Psi}(s) \neq 0$ for s in a subset of $[t_0,t_1^*]$ of positive measure. This completes the proof of the theorem.

We now wish to formulate a control problem with terminal manifold in a function space and a type of "bounded state variable" constraint. To simplify expressions we assume the final time, T, is fixed. Let $h \in [0,T-t_0]$; $L_{-\mu},\ldots,L_0,\ldots,L_m$ be given real-valued, C^1 functions defined on $C([\alpha_0,t_0],\mathcal{G}) \times C([-h,0],\mathcal{G})$; π be a closed subset of $[t_0,T]$, $\tilde{g}$ be a real-valued function defined on $C([\alpha_0,T],R^n) \times \pi$, $\tilde{g}$ and $d\tilde{g}$ be continuous, where $d\tilde{g}$ is the Frechet differential of $\tilde{g}$ with respect to x; $\mathcal{F}$ be a quasi-convex family; $\Phi \subset C([\alpha_0,t_0],\mathcal{G})$; and $D(x(\cdot),t)$ satisfy (2.3)-(2.5) and (3.1). Define

$$\mathcal{E} = \{x \in Q(T)\},\quad Q(T) \text{ given by (3.2)},$$

$$\mathcal{Y}_0 = C(\pi,R),$$

$$Z_0 = \{y \in \mathcal{Y}_0 : y(t) < 0 \text{ for all } t \in \pi\}.$$

Z_0 is a non-empty, open convex cone. Also,

$$\overline{Z}_0 = \{y \in \mathcal{Y}_0 : y(t) \leq 0 \text{ for all } t \in \pi\}.$$

Let the functions $\varphi_i : \mathcal{E} \to R$, $i = -\mu,\ldots,m$, be given by

$$\varphi_i(x) = L_i(x_{t_0}, x_{T-h,T}),$$

where $x_{T-h,T} \in C([-h,0],R^n)$ is given by $x_{T-h,T}(\theta) = x(T+\theta)$. Let the func-

tion $\varphi_{-\mu-1}: \mathscr{E} \to \mathscr{Z}_0$ be given by

$$[\varphi_{-\mu-1}(x)](t) = \tilde{g}(x(\cdot),t) \quad \text{for all} \quad t \in \pi.$$

Problem 4.2: We wish to find $z \in \mathscr{E}$ such that

a) $\varphi_i(z) \leq 0$ for $i = -\mu,\ldots,-1$,

b) $\varphi_i(z) = 0$ for $i = 1,\ldots,m$,

c) $\varphi_{-\mu-1}(z) \in \overline{Z}_0$,

d) $\varphi_0(z) \leq \varphi_0(x)$ for all $x \in \mathscr{E}$ which satisfy a) - c).

For such a z, let φ^*, f^*, and Y be given by (3.3), (3.4), and the remarks after (3.4).

Theorem 4.2. Given the assumptions above and $\mu(\cdot,s)$ is of bounded variation on every bounded interval, $s \in [\alpha_0,T]$, let z be a solution of problem 4.2 such that there is at least one $t' \in \pi$ with $\tilde{g}(z(\cdot),t') = 0$, and either $\Phi = \{\varphi^*\}$ or $\varphi^* \in$ interior Φ. Then there exists a row n-vector valued function $\overline{\psi}$ defined on $[t_0,\infty)$, a real-valued function λ defined on R, and real numbers α^i, $i = -\mu,\ldots,m$, such that

i) $\lambda(s)$ is a non-increasing function of bounded variation, continuous from the right, $\lambda(T) = 0$, and λ is constant on each interval of

$$R \backslash \{t' \in \pi: \tilde{g}(z(\cdot),t') = 0\}.$$

ii) $\alpha^i \leq 0$ for $i \leq 0$, $\alpha^i = 0$ for all $i \in \{-\mu,\ldots,-1\}$ such that $\varphi_i(z) < 0$, $\sum_{i=-\mu}^{m} |\alpha^i| + |\lambda(t_0^-)| > 0$.

iii) $\overline{\psi}$ is given by

(4.16) $$\overline{\psi}(s) = \Sigma_{i=-\mu}^{m}\alpha^{i}dL_{i}[z_{t_0},z_{T-h,T};0,Y_{T-h,T}(s,\cdot)] + \int_{t_0}^{T}{}_{-}\, d\tilde{g}[z(\cdot),t;Y(s,\cdot)]d\lambda(t),$$

and satisfies

(4.17) $$\overline{\psi}(s) = 0, \quad s > T,$$

(4.18) $$\overline{\psi}(T) = \Sigma_{i=-\mu}^{m}\alpha^{i}dL_{i}[z_{t_0},z_{T-h,T};0,Y_{T-h,T}(T,\cdot)] + d\tilde{g}[z(\cdot),T;Y(T,\cdot)]\{\lambda(T)-\lambda(T-)\},$$

(4.19) $$\overline{\psi}(s) = \Sigma_{i=-\mu}^{m}\alpha^{i}dL_{i}[z_{t_0},z_{T-h,T};0,E^{s}_{T-h,T}] + \int_{s}^{T}{}_{-}\, d\tilde{g}[z(\cdot),t;E^{s}(\cdot)]d\lambda(t) + \int_{s}^{T^+} d_{\alpha}[\overline{\psi}(\alpha)]\mu(\alpha,s) - \int_{s}^{T}\overline{\psi}(\alpha)\eta^*(\alpha,s)d\alpha, \quad s \in [t_0,T),$$

where

$$E^{s}(t) = \begin{cases} 0, & t < s \\ E, & t \geq s \end{cases}$$

(4.20) $$\overline{\psi}(t_0) = -\Sigma_{i=-\mu}^{m}\alpha^{i}dL_{i}[z_{t_0},z_{T-h,T};Y_{t_0}(t_0,\cdot),0]$$

(note: (4.20) need not hold if $\Phi = \{\varphi^*\}$). If, in addition, $\Sigma_{i=-\mu}^{m}\beta^{i}dL_{i}[z_{t_0},z_{T-h,T};\cdot,\cdot] = 0$ and $\beta^{i} \leq 0$ for $i \in \{-\mu,\ldots,0\}$ implies

$\beta^i = 0$, $i = -\mu,\ldots,m$; and there do not exist β^i, $i = -\mu,\ldots,m$, $\beta^i \leq 0$ for $i \in \{-\mu,\ldots,0\}$ such that, for every $y \in C([\alpha_0,T],R^n)$,

$$\sum_{i=-\mu}^{m} \beta^i dL_i[z_{t_0}, z_{T-h,T}; y_{t_0}, y_{T-h,T}] + \int_{t_0}^{T} d\tilde{g}[z(\cdot),t;y(\cdot)]dt = 0;$$

then $\overline{\psi}$ is non-zero on a subset of $[t_0,T]$ of positive measure.

iv) the following maximum condition holds:

$$\int_{t_0}^{T} \overline{\psi}(s)f(z(\cdot),s)ds \leq \int_{t_0}^{T} \overline{\psi}(s)f^*(z(\cdot),s)ds \quad \text{for all} \quad f \in [\mathscr{F}].$$

Proof: Since z is a solution, it is a (φ,Φ,Z) extremal, where

$$\varphi = (\varphi_1,\ldots,\varphi_m),$$

$$\Phi = (\varphi_0-\varphi_0(z),\ \varphi_{-1},\ldots,\varphi_{-\mu},\varphi_{-\mu-1}),$$

$$\mathscr{Z} = R^{\mu+1} \times \mathscr{Z}_0,$$

$$Z = \{\gamma \in R^{\mu+1}:\ \gamma_j < 0,\ j = 1,\ldots,\mu+1\} \times Z_0.$$

Let $\mathscr{E}_1 = \mathscr{E}$, $\mathscr{Y} = C([\alpha_0,T],R^n)$. Clearly condition 6.1 is satisfied. By theorem 3.1 above, condition 6.2 is satisfied with $M = \mathscr{M}$ given by (3.7) and Θ given by theorem 3.1. Fix $x \in \mathscr{Y}$.

$$\begin{aligned} \varphi_{-\mu-1}(z+\mathcal{E}y)(t) - \varphi_{-\mu-1}(z)(t) &= \tilde{g}(z(\circ)+\mathcal{E}y(\circ),t) - \tilde{g}(z(\circ),t) \\ &= \tilde{g}(z(\circ)+\mathcal{E}y(\circ),t) - \tilde{g}(z(\circ)+\mathcal{E}x(\circ),t) \\ &\quad + \tilde{g}(z(\circ)+\mathcal{E}x(\circ),t) - \tilde{g}(z(\circ),t) \end{aligned}$$

$$= d\tilde{g}[z(\cdot)+\varepsilon\{\theta_1(t,\varepsilon,y)y(\cdot) + [1-\theta_1(t,\varepsilon,y)]x(\cdot)\},t;$$

$$\varepsilon\{y(\cdot)-x(\cdot)\}]$$

$$+ \{d\tilde{g}[z(\cdot)+\varepsilon\theta_2(t,\varepsilon)x(\cdot),t;\varepsilon x(\cdot)]$$

$$- d\tilde{g}[z(\cdot),t;\varepsilon x(\cdot)]\} + d\tilde{g}[z(\cdot),t;\varepsilon x(\cdot)],$$

where $0 \leq \theta_1, \theta_2 \leq 1$. By the continuity of $d\tilde{g}$ and x, existence of a bounded neighborhood of x, and uniform continuity of z over $[\alpha_0,T]$, we have

$$\lim_{\substack{\varepsilon \to 0^+ \\ y \to x}} \frac{\varphi_{-\mu-1}(z+\varepsilon y)(t) - \varphi_{-\mu-1}(z)(t)}{\varepsilon} = d\tilde{g}[z(\cdot),t;x(\cdot)]$$

uniformly in $t \in \pi$. Using this and theorem 6.4 of [22], conditions 6.3 and 6.4 are satisfied with

(4.21) $$h(y) = (dL_1[z_{t_0},z_{T-h,T};y_{t_0},y_{T-h,T}],\dots,dL_m[z_{t_0},z_{T-h,T};y_{t_0},y_{T-h,T}]),$$

(4.22) $$\hat{h}(y) = (dL_0[z_{t_0},z_{T-h,T};y_{t_0},y_{T-h,T}],\dots,dL_{-\mu}[z_{t_0},z_{T-h,T};y_{t_0},y_{T-h,T}],$$
$$d\tilde{g}[z(\cdot),\cdot;y(\cdot)]).$$

Thus, by theorems 6.1 and 3.1 of [22], there exists a continuous linear functional $\ell \in \mathcal{Z}^*$ and a vector $\alpha \in R^m$ such that

(4.23) $$\alpha\cdot h(y) + \ell[\hat{h}(y)] \leq 0 \quad \text{for all } y \in M,$$

(4.24) $$\text{if } \alpha = 0, \text{ then } \ell \neq 0,$$

(4.25) $$\ell(\zeta) \geq 0 \text{ for all } \zeta \in \overline{Z},$$

(4.26) $$\ell[\Phi(z)] = 0.$$

Since $\mathcal{Z} = R^{\mu+1} \times \mathcal{Z}_0$, there is a vector $\hat{\alpha} = (\alpha^0, \alpha^{-1}, \ldots, \alpha^{-\mu}) \in R^{\mu+1}$, and a continuous linear functional $\ell_{-\mu-1} \in \mathcal{Z}_0^*$ such that

(4.27) $$\ell[(\gamma,y)] = \hat{\alpha}\cdot\gamma + \ell_{-\mu-1}[y] \text{ for each } (\gamma,y) \in R^{\mu+1} \times \mathcal{Z}_0.$$

Since $(-e_i,0) \in \overline{Z}$ and $(0,y) \in \overline{Z}$ whenever e_i is a standard basis vector of $R^{\mu+1}$ and $y \in \overline{Z}_0$, from (4.25)-(4.27) we have

(4.28) $$\ell_{-\mu-1}[y] \geq 0 \text{ whenever } y \in \overline{Z}_0.$$

(4.29) $$\alpha^i \leq 0,\ i = 0,-1,\ldots,-\mu, \text{ and } \alpha^i = 0 \text{ for all }$$
$$i \in \{-1,\ldots,-\mu\} \text{ such that } \varphi_i(z) < 0.$$

(4.30) $$\ell_{-\mu-1}[\tilde{g}(z(\cdot),\cdot)] = 0.$$

Also, by (4.21)-(4.23) and (4.27),

(4.31) $$\Sigma_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0}, z_{T-h,T}; y_{t_0}, y_{T-h,T}] + \ell_{-\mu-1}\{d\tilde{g}[z(\cdot),\cdot;y(\cdot)]\} \leq 0$$
$$\text{for all } y \in M.$$

By the representation of elements of $\mathscr{G}_0^*$, there exists a function $\lambda\colon R \to R$ of bounded variation, continuous from the right, constant on each sub-interval of $R\backslash\pi$, $\lambda(T) = 0$, such that

$$\ell_{-\mu-1}[y] = \int_{t_0^-}^{T} y(t)d\lambda(t) \quad \text{for all} \quad y \in \mathscr{G}_0. \tag{4.32}$$

(4.28) implies that λ is non-increasing. From λ non-increasing and (4.30), λ is constant on each interval J of $[t_0,T]$ such that $\tilde{g}(z(\cdot),t) < 0$ for all $t \in J \cap \pi$. Thus i) is proved. From (4.24), (4.27), and (4.32), $\sum_{i=-\mu}^{m}|\alpha^i| + |\lambda(t_0^-)| > 0$. Together with (4.29), this is ii). From (3.7), (4.31), (4.32), for all $\delta\varphi \in \mathscr{P}-\varphi^*$ and $\delta f \in [\mathscr{F}]-f^*$,

$$\begin{aligned} &\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0}, z_{T-h,T}; \delta\varphi, \delta x_{T-h,T}(\cdot;\delta\varphi,\delta f)] \\ &\qquad + \int_{t_0^-}^{T} d\tilde{g}[z(\cdot),t;\delta x(\cdot;\delta\varphi,\delta f]d\lambda(t) \le 0. \end{aligned} \tag{4.33}$$

Let $\delta\varphi = 0$, $\delta f \in [\mathscr{F}]-f^*$. Then from (2.11), (3.5), and (4.33),

$$\begin{aligned} &\sum_{i=-\mu}^{m}\alpha^i dL_i[z_{t_0}, z_{T-h,T}; 0, \int_{t_0}^{T} Y_{T-h,T}(s,\cdot)\delta f(z(\cdot),s)ds] \\ &\qquad + \int_{t_0^-}^{T} d\tilde{g}[z(\cdot),t; \int_{t_0}^{T} Y(s,\cdot)\delta f(z(\cdot),s)ds]d\lambda(t) \le 0. \end{aligned}$$

By lemma 2.3, Y is Borel-measurable. Noting that the dL_i and $d\tilde{g}$ represent integrations, we apply Fubini's Theorem to interchange orders of integration. Thus, for all $\delta f \in [\mathscr{F}]-f^*$,

$$\int_{t_0}^{T} \sum_{i=-\mu}^{m} \alpha^{i} dL_{i}[z_{t_0}, z_{T-h,T}; 0, Y_{T-h,T}(s,\cdot)]\delta f(z(\cdot),s)ds$$
$$+ \int_{t_0}^{T} \{\int_{t_0^-}^{T} d\tilde{g}[z(\cdot),t;Y(s,\cdot)]d\lambda(t)\}\delta f(z(\cdot),s)ds \leq 0.$$

Using (4.16) to define $\overline{\Psi}(s)$, this is iv). (2.11) and (4.16) imply (4.17). By (2.11), (4.16) may be rewritten as

$$\overline{\Psi}(s) = \sum_{i=-\mu}^{m} \alpha^{i} dL_{i}[z_{t_0}, z_{T-h,T}; 0, Y_{T-h,T}(s,\cdot)]$$
$$+ \int_{s^-}^{T} d\tilde{g}[z(\cdot),t;Y(s,\cdot)]d\lambda(t),$$

which implies (4.18). Applying (4.16) to (2.9), then interchanging the orders of integration (justified by a theorem of Cameron and Martin [6]), we obtain (4.19).

Now let $\delta f = 0$. If $\varphi^* \in$ interior Φ, there exists $\rho > 0$ such that if $\mathscr{P}' = \{\varphi \in C([\alpha_0, t_0], R^n): \|\varphi - \varphi^*\| < \rho\}$, then $\mathscr{P}' \subset \mathscr{P}$. From (3.5), (4.33), and the fact that $-\delta\varphi \in \mathscr{P}' - \varphi^*$ if $\delta\varphi \in \mathscr{P}' - \varphi^*$; for all $\delta\varphi \in \mathscr{P}' - \varphi^*$,

$$\sum_{i=-\mu}^{m} \alpha^{i} dL_{i}[z_{t_0}, z_{T-h,T}; \delta\varphi, \cdot][Y_{T-h,T}(t_0,\cdot)D(\delta\varphi, t_0)$$
$$+ \int_{\alpha_0}^{t_0^-} d_{\sigma}\{-\int_{t_0}^{T^+} d_{\alpha}[Y_{T-h,T}(\alpha,\cdot)]\mu(\alpha,\sigma) + \int_{t_0}^{T} Y_{T-h,T}(\alpha,\cdot)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma)]$$
$$+ \int_{t_0^-}^{T} d\tilde{g}[z(\cdot),t;\cdot][Y(t_0,\cdot)D(\delta\varphi, t_0)$$
$$+ \int_{\alpha_0}^{t_0^-} d_{\sigma}\{-\int_{t_0}^{T^+} d_{\alpha}[Y(\alpha,\cdot)]\mu(\alpha,\sigma) + \int_{t_0}^{T} Y(\alpha,\cdot)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma)$$
$$+ \delta\varphi(\cdot;t)]d\lambda(t) = 0,$$

where we use the notation

$$\delta\varphi(s;t) = \begin{cases} \delta\varphi(s), & s \in [\alpha_0, t_0) \\ 0 \quad , & s \in [t_0, t]. \end{cases}$$

Interchanging the order of the multiple integrals (justified by [6]) and grouping terms,

$$\begin{aligned}
&\textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; \delta\varphi, 0] + \int_{t_0^-}^{T} d\tilde{g}[z(\cdot), t; \delta\varphi(\cdot;t)]d\lambda(t) \\
&+ \textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; 0, Y_{T-h,T}(t_0,\cdot)D(\delta\varphi, t_0)] \\
&\quad + \int_{t_0^-}^{T} d\tilde{g}[z(\cdot), t; Y(t_0,\cdot)D(\delta\varphi, t_0)]d\lambda(t) \\
&+ \int_{\alpha_0}^{t_0^-} d_\sigma \Bigg\{ -\int_{t_0}^{T^+} d_\alpha \Big[\textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; 0, Y_{T-h,T}(\alpha,\cdot)]\Big]\mu(\alpha,\sigma) \\
&\quad - \int_{t_0}^{T^+} d_\alpha \Big[\int_{t_0^-}^{T} d\tilde{g}[z(\cdot), t; Y(\alpha,\cdot)]d\lambda(t)\Big] \mu(\alpha,\sigma) \Bigg\}\delta\varphi(\sigma) \\
&+ \int_{\alpha_0}^{t_0^-} d_\sigma \Bigg\{ \int_{t_0}^{T} \textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; 0, Y_{T-h,T}(\alpha,\cdot)]\eta^*(\alpha,\sigma)d\alpha \\
&\quad + \int_{t_0}^{T} [\int_{t_0^-}^{T} d\tilde{g}[z(\cdot), t; Y(\alpha,\cdot)]d\lambda(t)]\eta^*(\alpha,\sigma)d\alpha \Bigg\}\delta\varphi(\sigma) = 0
\end{aligned}$$

Using (4.16), we obtain for all $\delta\varphi \in \mathcal{F}' - \varphi^*$,

$$\begin{aligned}
&\textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; \delta\varphi, 0] + \int_{t_0^-}^{T} d\tilde{g}[z(\cdot), t; \delta\varphi(\cdot;t)]d\lambda(t) \\
&\quad + \bar{\psi}(t_0)D(\delta\varphi, t_0)
\end{aligned}$$

$$+\int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{T^+} d_\alpha[\bar{\psi}(\alpha)]\mu(\alpha,\sigma) + \int_{t_0}^{T} \bar{\psi}(\alpha)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma) = 0.$$

Define $\delta\tilde{\varphi}_{\xi,\zeta}$ by (4.14). Substituting this in the last equation and passing to the limit as $\zeta \to t_0^-$,

$$\{\textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T}; Y_{t_0}(t_0,\cdot),0] + \bar{\psi}(t_0)\}\xi = 0$$

for all $\xi \in P$. Thus the quantity in braces vanishes, which is (4.20).

There remains only to show that, given the additional assumptions, $\bar{\psi}$ is non-zero on a subset of $[t_0,T]$ of positive measure. For each $x \in C([\alpha_0,T],R^n)$ with $\|x_{t_0}\| < \rho$, define w_x and φ_x as in the proof of theorem 4.1; again x is given by (4.15). Define the functional on $C([\alpha_0,T],R^n)$

$$\tilde{L}(\cdot) = \textstyle\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0}, z_{T-h,T};\cdot,\cdot] + \ell_{-\mu-1}\{d\tilde{g}[z(\cdot),t;\cdot]\}.$$

From (4.33) and the fact that $\delta\varphi \in \mathscr{P}'-\varphi^*$ implies $-\delta\varphi \in \mathscr{P}'-\varphi^*$, we obtain that, for all $\delta\varphi \in \mathscr{P}'-\varphi^*$, $\tilde{L}(\delta x(\cdot;\delta\varphi,0)) = 0$. Thus, from (4.15), (4.16), and (4.32), after changing orders of integration (justified by [6]), $\tilde{L}(x) = \int_{t_0}^{T} \bar{\psi}(s)w_x(s)ds$. By the additional assumptions and ii), $\tilde{L} \neq 0$, and so $\bar{\psi}(s) \neq 0$ for s in a subset of $[t_0,T]$ of positive measure. This completes the proof of the theorem.

Remark 4.1. If $\tilde{g}(z(\cdot),t) < 0$ for all $t \in \pi$, then $\ell_{-\mu-1} = 0$ and $\lambda(t) = 0$ for all $t \in R$. Hence $\int_{t_0^-}^{T} \ldots . d\lambda(t) = 0$ wherever it appears.

Remark 4.2. In problem 4.1, the second component of the domain of the restraint functions may be taken to be $C([-h,0],\mathscr{G})$ as in problem 4.2, rather than $\mathscr{G}$. Then in theorem 4.1 we must add the hypotheses that $\mu(\cdot,s)$ is of bounded variation on every bounded interval, and that $\dot{z}$ exists on $[t^*-h,t^*]$ and is bounded there (since $\dot{z}$ is clearly Borel-measurable, this will ensure that it is integrable with respect to each of the measures associated with $dL_i[z_{t_0},z_{t^*-h,t^*},t^*;0,\cdot,0]$, $i = -\mu,\ldots,m$). The following changes must be made in the theorem statement: In (4.1), (4.3), (4.6), and the additional assumption, $z(t^*)$ is replaced by z_{t^*-h,t^*}. In (4.1), $Y(s,t^*)$ is replaced by $Y_{t^*-h,t^*}(s,\cdot)$. In (4.3), E is replaced by $Y_{t^*-h,t^*}(t^*,\cdot)$. Equation (4.4) becomes

$$(4.4') \qquad \sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0},z_{t^*-h,t^*},t^*;0,\dot{z}_{t^*-h,t^*},0]$$

$$= -\sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0},z_{t^*-h,t^*},t^*;0,0,1].$$

Equation (4.5) becomes

$$(4.5') \qquad \bar{\Psi}(s) = \sum_{i=-\mu}^{m} \alpha^i dL_i[z_{t_0},z_{t^*-h,t^*},t^*;0,E^s_{t^*-h,t^*},0]$$

$$+ \int_s^{t^{*+}} d_\alpha[\bar{\Psi}(\alpha)]\mu(\alpha,s) - \int_s^{t^*} \bar{\Psi}(\alpha)\eta^*(\alpha,s)d\alpha,$$

where $E^s(t)$ is defined after (4.19).

This type of restraint would be quite natural, for example, if the

system were governed by $\frac{d}{dt} D(x_t,t) = f(x_t,u(t),t)$, $t \in [t_0,a)$, where $x_t \in C([-h,0], R^n)$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h,0]$. For such a system one also has $\alpha_0 = t_0-h$.

The following lemma is useful in problems with a fixed terminal function.

Lemma 4.1. Let $\tilde{g}\colon R^{n+1} \to R$ be such that $\tilde{g}(x,t)$ and $\tilde{g}_x(x,t)$ are continuous; $\tilde{I}(t_1)$ be a closed subset of R such that $\tilde{I}(t_1) = \tilde{I}(0) + t_1$ for $t_1 \in R$ and $\tilde{I}(t_0) \in [\alpha_0,t_0]$; $\varphi_{-\mu-1}(x,t_1) \in C(\tilde{I}(0),R)$ be given by $\varphi_{-\mu-1}(x,t_1)(\theta) = \tilde{g}(x(t_1+\theta),\theta)$, $\theta \in \tilde{I}(0)$. Given $(z,t^*) \in C([\alpha_0,a),R^n) \times [t_0,a)$ such that $\dot{z}$ exists and is continuous on a neighborhood of $\tilde{I}(t^*)$; then for all $x \in C([\alpha_0,a),R^n)$, $\tau \in R$,

$$\lim_{\substack{\varepsilon \to 0^+ \\ \sigma \to \tau \\ y \to x}} \frac{\varphi_{-\mu-1}(z+\varepsilon y,t^*+\varepsilon\sigma) - \varphi_{-\mu-1}(z,t^*)}{\varepsilon}(\theta)$$

$$= \tilde{g}_x(z(t^*+\theta),\theta)[x(t^*+\theta)+\dot{z}(t^*+\theta)\tau]$$

uniformly in $\theta \in \tilde{I}(0)$.

Proof: Fix $(x,\tau) \in C([\alpha_0,a),R^n) \times R^n$. For each $\theta \in \tilde{I}(0)$, if $t = t^*+\theta$,

$$\varphi_{-\mu-1}(z+\varepsilon y,t^*+\varepsilon\sigma)(\theta) = \tilde{g}(z(t+\varepsilon\sigma)+\varepsilon y(t+\varepsilon\sigma),\theta)$$

$$= \tilde{g}(z(t),\theta) + \tilde{g}(z(t)+\varepsilon x(t),\theta) - \tilde{g}(z(t),\theta)$$

$$+ \tilde{g}(z(t)+\varepsilon x(t+\varepsilon\sigma),\theta) - \tilde{g}(z(t)+\varepsilon x(t),\theta)$$

$$+ \tilde{g}(z(t)+\varepsilon y(t+\varepsilon\sigma),\theta) - \tilde{g}(z(t)+\varepsilon x(t+\varepsilon\sigma),\theta)$$

$$+ \tilde{g}(z(t+\varepsilon\sigma)+\varepsilon y(t+\varepsilon\sigma),\theta) - \tilde{g}(z(t)+\varepsilon y(t+\varepsilon\sigma),\theta)$$

$$
\begin{aligned}
&= \tilde{g}(z(t),\theta) + \tilde{g}_x(z(t)+\varepsilon\xi_1(\theta,\varepsilon)x(t),\theta)\varepsilon x(t) \\
&\quad + \tilde{g}_x(z(t)+\varepsilon\{\xi_2(\theta,\varepsilon,\sigma)x(t)+[1-\xi_2(\theta,\varepsilon,\sigma)]x(t+\varepsilon\sigma)\},\theta)\cdot \\
&\qquad\qquad \cdot\varepsilon[x(t+\varepsilon\sigma)-x(t)] \\
&\quad + \tilde{g}_x(z(t)+\varepsilon\{\xi_3(\theta,\varepsilon,\sigma,y)y(t+\varepsilon\sigma)+[1-\xi_3(\theta,\varepsilon,\sigma,y)]x(t+\varepsilon\sigma)\},\theta)\cdot \\
&\qquad\qquad \cdot\varepsilon[y(t+\varepsilon\sigma)-x(t+\varepsilon\sigma)] \\
&\quad + \tilde{g}_x(\{\xi_4(\theta,\varepsilon,\sigma,y)z(t+\varepsilon\sigma)+[1-\xi_4(\theta,\varepsilon,\sigma,y)]z(t)\}+\varepsilon y(t+\varepsilon\sigma),\theta)\cdot \\
&\qquad\qquad \cdot[z(t+\varepsilon\sigma)-z(t)],
\end{aligned}
$$

where $0 \leq \xi_i \leq 1$, $i = 1,2,3,4$. Thus, for each $\theta \in \tilde{I}(0)$,

$$
\frac{\varphi_{-\mu-1}(z+\varepsilon y,t^*+\varepsilon\sigma) - \varphi_{-\mu-1}(z,t^*)}{\varepsilon}(\theta) \tag{4.34}
$$

$$
= \tilde{g}_x(z(t^*+\theta),\theta)[x(t^*+\theta)+\mathring{z}(t^*+\theta)\tau] + \zeta(\theta;\varepsilon,\sigma,y),
$$

where

$$
\begin{aligned}
\zeta(\theta;\varepsilon,\sigma,y) &= [\tilde{g}_x(z(t)+\varepsilon\xi_1(\theta,\varepsilon)x(t),\theta) - \tilde{g}_x(z(t),\theta)]x(t) \\
&\quad + \tilde{g}_x(z(t)+\varepsilon\{\xi_2(\theta,\varepsilon,\sigma)x(t)+[1-\xi_2(\theta,\varepsilon,\sigma)]x(t+\varepsilon\sigma),\theta)\cdot \\
&\qquad\qquad \cdot[x(t+\varepsilon\sigma)-x(t)] \\
&\quad + \tilde{g}_x(z(t)+\varepsilon\{\xi_3(\theta,\varepsilon,\sigma,y)y(t+\varepsilon\sigma)+[1-\xi_3(\theta,\varepsilon,\sigma,y)]x(t+\varepsilon\sigma)\},\theta)\cdot \\
&\qquad\qquad \cdot[y(t+\varepsilon\sigma)-x(t+\varepsilon\sigma)] \\
&\quad +[\tilde{g}_x(\{\xi_4(\theta,\varepsilon,\sigma,y)z(t+\varepsilon\sigma)+[1-\xi_4(\theta,\varepsilon,\sigma,y)]z(t)\}+\varepsilon y(t+\varepsilon\sigma),\theta)
\end{aligned}
$$

$$-\tilde{g}_x(z(t),\theta)][\frac{z(t+\varepsilon\sigma)-z(t)}{\varepsilon}]$$

$$+\ \tilde{g}_x(z(t),\theta)[\psi(t,\varepsilon,\sigma)-\mathring{z}(t)]\sigma$$

$$+\ \tilde{g}_x(z(t),\theta)\mathring{z}(t)[\sigma-\tau],$$

and the vector ψ is given by the components $\psi_i(t,\varepsilon,\sigma) = \mathring{z}_i(t+\varepsilon\xi_{4+i}(\varepsilon,t,\sigma)\sigma)$, $i = 1,2,\ldots,n$, with $0 \leq \xi_{4+i} \leq 1$, $i = 1,2,\ldots,n$. Since $\tilde{g}$, $\tilde{g}_x$, z, and x are continuous, and $\mathring{z}$ is continuous on a neighborhood of $\tilde{I}(t^*)$, we have that

$$\lim_{\substack{\varepsilon\to 0^+\\ \sigma\to\tau\\ y\to x}} \|\zeta(\cdot;\varepsilon,\sigma,y)\| = 0.$$

This and (4.34) prove the lemma.

Lemma 4.2. Let $\tilde{g}$, $\tilde{I}$ be as in lemma 4.1, $\varphi_{-\mu-1}(x,t_1) = \sup_{\theta\in\tilde{I}(0)} \tilde{g}(x(t_1+\theta),\theta)$. Given (z,t^*) as in lemma 4.1 such that $\varphi_{-\mu-1}(z,t^*) = 0$; then for all $x \in C([\alpha_0,a),R^n)$, $\tau \in R$,

$$\lim_{\substack{\varepsilon\to 0^+\\ \sigma\to\tau\\ y\to x}} \frac{\varphi_{-\mu-1}(z+\varepsilon y,t^*+\varepsilon\sigma) - \varphi_{-\mu-1}(z,t^*)}{\varepsilon}$$

$$= \sup_{\theta\in\tilde{I}(0)} \{\tilde{g}_x(z(t^*+\theta),\theta)[x(t^*+\theta)+\mathring{z}(t^*+\theta)\tau]\}.$$

Proof: The proof is essentially the same as the proof of lemma 4.4.

Remark 4.3. In both lemmas, if $t^* = T$ is fixed, the term $\mathring{z}(t^*+\theta)\tau$ does not appear and it is no longer necessary to require that $\mathring{z}$ exist on a neighborhood of $\tilde{I}(t^*)$.

We now formulate a control problem with a fixed function as target. Let $h > 0$ be fixed; $\zeta\colon [-h,0] \to R^n$ be C^1 (see remark 4.5 below), $\zeta \not\equiv 0$; W be a diagonal matrix whose entries are either 0 or 1, such that $W\zeta = \zeta$; L_0 be a real-valued, C^1 function defined on $C([\alpha_0,t_0],R^n) \times C([-h,0],R^n) \times [t_0,a)$; $\mathscr{F}$ be a quasi-convex family; $\Phi \subset C([\alpha_0,t_0],\mathscr{G})$; $D(x(\cdot),t)$ satisfy (2.3) - (2.5) and (3.1); $\mathscr{L} = \{(x,t)\colon t \in [t_0,a),\ x \in Q(t)\}$, where $Q(t)$ is given by (3.2); $\mathscr{Z}_0 = C([-h,0],R)$; and $Z_0 = \{y \in \mathscr{Z}_0\colon y(\theta) < 0$ for $\theta \in [-h,0]\}$. Define the function $\varphi_0\colon \mathscr{L} \to R$ by

$$\varphi_0(x,t_1) = L_0(x_{t_0}, x_{t_1-h,t_1}, t_1),$$

and the functions $\varphi_i\colon \mathscr{L} \to \mathscr{Z}_0$, $i = -1,-2$, by

$$\varphi_{-1}(x,t_1)(\theta) = \zeta(\theta)[\zeta(\theta) - Wx(t_1+\theta)],$$
$$\varphi_{-2}(x,t_1)(\theta) = [Wx(t_1+\theta)]^2 - [\zeta(\theta)]^2.$$

Problem 4.3. We wish to find $(z,t^*) \in \mathscr{L}$ such that

a) $\varphi_i(z,t^*) \in \bar{Z}_0$, $i = -1,-2$,

b) $\varphi_0(z,t^*) \le \varphi_0(x,t_1)$ for all $(x,t_1) \in \mathscr{L}$ which satisfy a).

For such a (z,t^*), let φ^*, f^*, and Y be given by (3.3), (3.4), and the remarks after (3.4).

Remark 4.4. We note that for any (x,t_1) satisfying a), $Wx_{t_1-h,t_1} = \zeta$. To illustrate the use of such a W, where $W \neq E$, suppose we have the following problem in $C([\alpha_0,T],R^{n-1})$:

$$\bar{x}_{t_0} = \bar{\varphi}, \ \frac{d}{dt}\{\bar{x}(t) - \int_{\alpha_0}^{t} d_\theta[\bar{\mu}(t,\theta)]\bar{x}(\theta)\} = \bar{f}(\bar{x}(\cdot),u(t),t).$$

We wish to control $\bar{x}$ so that $\bar{x}(t) = \bar{\zeta}(t)$ on $[T-h,T]$ and the cost $J(u) = \int_{t_0}^{T} f^n(\bar{x}(\cdot),u(s),s)ds$ is minimized. Assume the appropriate hypotheses are satisfied, and f^n satisfies the same hypotheses as $\bar{f}$. We then set

$$x = \begin{pmatrix} \dot{\bar{x}} \\ x^n \end{pmatrix}, \ x_{t_0} = \begin{pmatrix} \bar{\varphi} \\ 0 \end{pmatrix},$$

$$\mu(t,\theta) = \begin{pmatrix} \bar{\mu}(t,\theta) & 0 \\ 0 & 0 \end{pmatrix}, \quad f(x(\cdot),u(s),s) = \begin{pmatrix} \bar{f}(\bar{x}(\cdot),u(s),s) \\ f^n(\bar{x}(\cdot),u(s),s) \end{pmatrix},$$

$\zeta = \begin{pmatrix} \bar{\zeta} \\ 0 \end{pmatrix}$, $W = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ where I is the $(n-1) \times (n-1)$ identity matrix,

$L_0(x_{t_0},x_{T-h,T}) = x^n(T)$. The augmented problem is in the form of problem 4.3.

Theorem 4.3. Given the assumptions above and, for every $s \in [\alpha_0,a)$, $\mu(\cdot,s)$ is of bounded variation on every bounded interval; let (z,t^*) be a solution of problem 4.3 such that $\dot{z}$ exists on $[t^*-h,t^*]$ and is bounded there, $W\dot{z}$ exists and is continuous on $(t^*-h-\beta, t^*+\beta)$ for some $\beta > 0$, and either $\Phi = \{\varphi^*\}$ or $\varphi^* \in$ interior Φ. Then there exist a row n-vector valued function ψ defined on $[t_0,\infty)$, a real-valued function $\lambda = 2\lambda_2 - \lambda_1$ defined on R, and a real number α^0, such that

i) λ_i is a non-increasing function of bounded variation, continuous from the left, $\lambda_i(0^+) = 0$, λ_i constant on $(-\infty,-h]$ and on $(0,\infty)$, $i = 1,2$,

$$\alpha^0 \leq 0, \quad |\alpha^0| + \operatorname*{var}_{[-h,\infty)} \lambda_1 + \operatorname*{var}_{[-h,\infty)} \lambda_2 > 0.$$

ii) ψ is given by

$$\psi(s) = \alpha^{0} dL_{0}[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, Y_{t^*-h,t^*}(s,\cdot), 0] + \int_{-h}^{0^+} \zeta(\theta) Y(s, t^*+\theta) d\lambda(\theta), \tag{4.35}$$

and satisfies

$$\psi(s) = 0 \quad \text{for} \quad s > t^*, \tag{4.36}$$

$$\psi(t^*) = \alpha^{0} dL_{0}[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, Y_{t^*-h,t^*}(t^*,\cdot), 0] - \zeta(0)\lambda(0), \tag{4.37}$$

$$\psi(s) = \alpha^{0} dL_{0}[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, E^{s}_{t^*-h,t^*}, 0] + \int_{-h}^{0^+} \zeta(\theta) E^{s}(t^*+\theta) d\lambda(\theta) + \int_{s}^{t^{*+}} d_{\alpha}[\psi(\alpha)]\mu(\alpha, s) - \int_{s}^{t^*} \psi(\alpha)\eta^*(\alpha, s) d\alpha \tag{4.38}$$

for $s \in [t_0, t^*)$, where $E^{s}(t)$ is defined after (4.19),

$$\alpha^{0} dL_{0}[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, \dot{z}_{t^*-h,t^*}, 1] + \int_{-h}^{0^+} \zeta(\theta)\dot{\zeta}(\theta) d\lambda(\theta) = 0, \tag{4.39}$$

$$\psi(t_0) = -\alpha^{0} dL_{0}[z_{t_0}, z_{t^*-h,t^*}, t^*; Y_{t_0}(t_0,\cdot), 0, 0] \tag{4.40}$$

iii) the following maximum condition holds:

$$\int_{t_0}^{t^*} \psi(s)f(z(\cdot),s)ds \leq \int_{t_0}^{t^*} \psi(s)f^*(z(\cdot),s)ds \quad \text{for all} \quad f \in \mathscr{F} .$$

Remark 4.5. If $t^* = T$ is fixed, (4.39) does not appear and it is no longer necessary to assume that $\mathring{\zeta}$ and $\dot{z}$ exist. In most cases the derivatives will exist, and the effect of fixing t^* is to remove the requirements that ζ and $W\dot{z}$ be continuous and $\dot{z}$ be bounded. If $\Phi = \{\varphi^*\}$, (4.40) need not hold.

Proof: Since (z,t^*) is a solution, it is a (Φ,Z) extremal, where

$$\Phi = (\varphi_0 - \varphi_0(z,t^*),\ \varphi_{-1},\varphi_{-2}),$$

$$\mathscr{Y} = R \times \mathscr{Z}_0 \times \mathscr{Z}_0,$$

$$Z = \{\gamma \in R\colon \gamma < 0\} \times Z_0 \times Z_0.$$

Choose $\varepsilon \in (0,a-t^*)$ such that $z \in Q(t^*+\varepsilon)$, (such an ε exists by theorem 2.3) and set $t' = t^*+\varepsilon$. Define $\mathscr{S}_1 = Q(t') \times [t_0,t']$, $\mathscr{Y} = C([\alpha_0,t'],R^n) \times R$. Clearly condition 6.1 is satisfied. By theorem 3.1 above, condition 6.2 is satisfied by $M = \mathscr{M} \times [-\varepsilon,\varepsilon]$, where $\mathscr{M}$ is given by (3.7). Theorem 6.4 in [22] implies condition 6.4 is satisfied for φ_0 when

$$h_0(y,\tau) = dL_0[z_{t_0},z_{t^*-h,t^*},t^*;y_{t_0},y_{t^*-h,t^*}+\mathring{z}_{t^*-h,t^*}\tau,\tau]. \tag{4.41}$$

By lemma 4.1, condition 6.4 is satisfied for φ_{-1} when

$$h_{-1}(y,\tau)(\theta) = -\zeta(\theta)[Wy(t^*+\theta) + W\mathring{z}(t^*+\theta)\tau],$$

and for φ_{-2} when

$$h_{-2}(y,\tau)(\theta) = 2Wz(t^*+\theta)[Wy(t^*+\theta) + W\mathring{z}(t^*+\theta)\tau].$$

Since $Wz_{t^*-h,t^*} = \zeta$, $W\mathring{z}_{t^*-h,t^*} = \mathring{\zeta}$, and $\zeta W = \zeta$, these are

(4.42) $$h_{-1}(y,\tau)(\theta) = -\zeta(\theta)[y(t^*+\theta) + \mathring{\zeta}(\theta)\tau],$$

(4.43) $$h_{-2}(y,\tau)(\theta) = 2\zeta(\theta)[y(t^*+\theta) + \mathring{\zeta}(\theta)\tau].$$

By theorems 6.2 and 3.1 of [22], there exists a non-zero $\ell \in \mathscr{Z}^*$ such that, if $\hat{h} = (h_0, h_{-1}, h_{-2})$,

(4.44) $$\ell[\hat{h}(y,\tau)] \leq 0 \quad \text{for all} \quad (y,\tau) \in M,$$

(4.45) $$\ell[x] \geq 0 \qquad \text{for all} \quad x \in \overline{Z}.$$

Since $\mathscr{Z} = R \times \mathscr{Z}_0 \times \mathscr{Z}_0$, there are an $\alpha^0 \in R$ and functionals ℓ_1, $\ell_2 \in \mathscr{Z}_0^*$ such that

$$\ell(\gamma,\omega,\xi) = \alpha^0\gamma + \ell_{-1}(\omega) + \ell_2(\xi) \quad \text{for each} \quad (\gamma,\omega,\xi) \in \mathscr{Z}.$$

By the representation of $\mathscr{Z}_0^*$, there exist functions $\lambda_i\colon R \to R$ of bounded variation, continuous from the left, $\lambda_i = 0$ on $(0,\infty)$, λ_i constant on $(-\infty,-h]$, such that for $\omega \in \mathscr{Z}_0$,

$$\ell_i(\omega) = \int_{-h}^{0^+} \omega(\theta)d\lambda_i(\theta), \quad i = 1,2.$$

Substituting the above and (4.41)-(4.43) in (4.44) and (4.45), we obtain

$$(4.46) \quad \alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; y_{t_0}, y_{t^*-h,t^*} + \mathring{z}_{t^*-h,t^*}\tau, \tau] - \int_{-h}^{0^+} \zeta(\theta)[y(t^*+\theta) + \mathring{\zeta}(\theta)\tau]d\lambda_1(\theta) + 2\int_{-h}^{0} \zeta(\theta)[y(t^*+\theta)+\mathring{\zeta}(\theta)\tau]d\lambda_2(\theta) \leq 0 \quad \text{for all} \quad (y,\tau) \in M,$$

$$(4.47) \quad \alpha^0\gamma + \int_{-h}^{0^+} \omega(\theta)d\lambda_1(\theta) + \int_{-h}^{0^+} \xi(\theta)d\lambda_2(\theta) \geq 0 \quad \text{for all} \quad (\gamma,\omega,\xi) \in \overline{Z}.$$

Since $(-1,0,0)$, $(0,\omega,0)$, $(0,0,\omega) \in \overline{Z}$ if $\omega \in \overline{Z}_0$, from (4.47), $\alpha^0 \leq 0$ and the λ_i are increasing. ℓ is non-zero, hence $|\alpha^0| + \operatorname{var}_{[-h,\infty)} \lambda_1 + \operatorname{var}_{[-h,\infty)} \lambda_2 > 0$. Thus i) has been proved. From the definition of M and (3.7), for all $\delta\varphi \in \mathscr{P}-\varphi^*$, $\delta f \in [\mathscr{F}]-f^*$, and $\tau \in [-\mathcal{E},\mathcal{E}]$, letting $\lambda(\theta) = 2\lambda_2(\theta) - \lambda_1(\theta)$, (4.46) is

$$(4.48) \quad \alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; \delta\varphi, \delta x_{t^*-h,t^*}(\cdot;\delta\varphi,\delta f) + \mathring{z}_{t^*-h,t^*}\tau, \tau] + \int_{-h}^{0^+} \zeta(\theta)[\delta x(t^*+\theta;\delta\varphi,\delta f) + \mathring{\zeta}(\theta)\tau]d\lambda(\theta) \leq 0.$$

Let $\delta\varphi = 0$, $\tau = 0$; by (3.5) and (4.48), for all $\delta f \in [\mathscr{F}]-f^*$,

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, \int_{t_0}^{t^*} Y_{t^*-h,t^*}(s,\cdot)\delta f(z(\cdot),s)ds, 0] + \int_{-h}^{0^+} \zeta(\theta)\int_{t_0}^{t^*} Y(s,t^*+\theta)\delta f(z(\cdot),s)ds\, d\lambda(\theta) \leq 0.$$

By lemma 2.3, Y is Borel-measurable. Interchanging dL_0 and the integral in the second term, and defining ψ by (4.35), this is iii). (4.36) and (4.37) follow from (2.10), (2.11), and (4.35). Applying (4.35) to (2.9), then changing orders of integration (justified by [6]), we obtain (4.38).

Let $\delta\varphi = 0$, $\delta f = 0$, $\tau \in [-\varepsilon,\varepsilon]$; then (4.48) is

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, \mathring{z}_{t^*-h,t^*}, \tau, \tau]$$

$$+ \int_{-h}^{0^+} \zeta(\theta)\mathring{\zeta}(\theta)\tau d\lambda(\theta) \leq 0.$$

Since τ is symmetric about zero, this implies (4.39).

Now let $\delta f = 0$, $\tau = 0$. If $\varphi^* \in$ interior Φ, there exists $\rho > 0$ such that if $\mathscr{P}' = \{\varphi \in C([\alpha_0, t_0], R^n): \|\varphi-\varphi^*\| < \rho\}$, then $\mathscr{P}' \subset \mathscr{P}$. From (3.5), (4.48), and the fact that $-\delta\varphi \in \mathscr{P}'-\varphi^*$ if $\delta\varphi \in \mathscr{P}'-\varphi^*$; for all $\delta\varphi \in \mathscr{P}'-\varphi^*$

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; \delta\varphi, 0, 0]$$

$$+ \alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, \cdot, 0]\Big[Y_{t^*-h,t^*}(t_0,\cdot)D(\delta\varphi, t_0)$$

$$+ \int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^{*+}} d_\alpha[Y_{t^*-h,t^*}(\alpha,\cdot)]\mu(\alpha,\sigma) + \int_{t_0}^{t^*} Y_{t^*-h,t^*}(\alpha,\cdot)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma)\Big]$$

$$+ \int_{-h}^{0^+} \zeta(\theta)\Big[Y(t_0, t^*+\theta)D(\delta\varphi, t_0) + \int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^{*+}} d_\alpha[Y(\alpha, t^*+\theta)]\mu(\alpha,\sigma)$$

$$+ \int_{t_0}^{t^*} Y(\alpha, t^*+\theta)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma)\Big]d\lambda(\theta) = 0.$$

Changing some orders of integration and using (4.35) this is

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; \delta\varphi, 0, 0] + \psi(t_0)D(\delta\varphi, t_0)$$

$$+ \int_{\alpha_0}^{t_0^-} d_\sigma\{-\int_{t_0}^{t^{*+}} d_\alpha[\psi(\alpha)]\mu(\alpha,\sigma) + \int_{t_0}^{t^*} \psi(\alpha)\eta^*(\alpha,\sigma)d\alpha\}\delta\varphi(\sigma) = 0.$$

Define $\delta\tilde{\varphi}_{\xi,\nu}$ by (4.14). Substituting this in the last equation and passing to the limit as $\nu \to t_0^-$,

$$\{\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; Y_{t_0}(t_0,\cdot), 0, 0] + \psi(t_0)\}\xi = 0$$

for all $\xi \in P$. Thus the quantity in braces vanishes which is (4.40), and the proof is complete.

Remark 4.6. The conditions of theorem 4.3 will be satisfied if $\alpha^0 = 0$, λ_2 is any function satisfying the restrictions in i), and $\lambda_1 = 2\lambda_2$. The examples in chapter 6 will show that there are cases in which a non-trivial maximum principle is given by theorem 4.3; but they also require that $\zeta(\theta) \neq 0$ at points θ where λ has a jump, further restricting the class of ζ which can be used as target functions.

Remark 4.7. Problem 4.3 may be expanded to include more restraint functions L_i as in problems 4.1 and 4.2. The corresponding dL_i will then appear in ψ as in theorems 4.1 and 4.2. The additional L_i may be used to restrain coordinates of x not specified by ζ, as well as to define an initial manifold in $C([\alpha_0, t_0], \mathscr{G})$.

Remark 4.8. The general method of proof used above can be applied to obtain necessary conditions for a wide variety of constraint functions. In particular, there are many functions which do not have a Frechet derivative, yet satisfy the requirements of [22] (see [21] and [22] for examples).

Remark 4.9. Theorems 4.2 and 4.3 use the same method of treating the bounded state variable constraint based on $\tilde{g}$. In [21, section 8], Neustadt also gives a method of treating bounded state variable constraints by a supremum function This treatment of problem 4.3 is presented below for completeness; the non-linear maximum principle it produces is very difficult to work with. In addition, due to the non-linearity, although a pointwise maximum principle is implied by the integral maximum principle, one can give examples where the pointwise maximum principle does not imply the integral maximum principle.

Let h, ζ, W, L_0, $\mathscr{F}$, Φ, D, and $\mathscr{L}$ be as in problem 4.3. Define the functions $\varphi_i : \mathscr{L} \to R$, $i = -2,-1,0$, by

$$\varphi_0(x,t_1) = L_0(x_{t_0}, x_{t_1-h}, t_1, t_1),$$

$$\varphi_{-1}(x,t_1) = \int_{-h}^{0} \zeta(\theta)[\zeta(\theta) - Wx(t_1+\theta)]d\theta,$$

$$\varphi_{-2}(x,t_1) = \sup_{\theta\in[-h,0]} \{[Wx(t_1+\theta)]^2 - [\zeta(\theta)]^2\}.$$

Problem 4.4: We wish to find $(z,t^*) \in \mathscr{L}$ such that

a) $\varphi_i(z,t^*) \leq 0$, $i = -2,-1$

b) $\varphi_0(z,t^*) \leq \varphi_0(x,t_1)$ for all $(x,t_1) \in \mathscr{L}$ which satisfy a).

For such a (z,t^*), let φ^*, f^* and Y be given by (3.3), (3.4), and the remarks after (3.4).

Theorem 4.4. Assume the conditions above; μ satisfies the conditions of lemma 2.6 and lemma 2.7; either $p = 1$ or $h_\ell(t) = t-d_\ell$, $\ell = 1,\ldots,p$; $\nu(\cdot,s)$ is of bounded variation on bounded intervals for each $s \in [\alpha_0,a)$; and for each $T \in [t_0,a)$, $\tau \in [\alpha_0,T]$, $\mathcal{E} > 0$, there exists $\rho(\mathcal{E},\tau,T) > 0$ such that $|t-\tau| < \rho(\mathcal{E},\tau,T)$ implies $|\nu(s,t)-\nu(s,\tau)| < \mathcal{E}$ for all $s \in [t_0,T]$. Let (z,t^*) be a solution to the problem such that $\dot{z}$ exists on $[t^*-h,t^*]$ and is bounded there, $W\dot{z}$ exists and is continuous on $(t^*-h-\beta,t^*+\beta)$ for some $\beta > 0$, either $\Phi = \{\varphi^*\}$ or $\varphi^* \in$ interior Φ, and for all $w \in BV([\alpha_0,a),R^n)$, $t \in [t_0,a)$, the map $s \to \int_s^t w(\alpha)\eta^*(\alpha,s)d\alpha$ is continuous on $[t_0,a)$. Then there exist functions $\psi\colon [t_0,\infty) \to R^{n_*}$ and $\psi_1\colon [t_0,\infty) \times [-h,0] \to R^{n_*}$, and real numbers $\alpha^0,\alpha^{-1},\alpha^{-2}$ such that

i) $\alpha^i \leq 0$, $i = 0,-1,-2$; $\sum_{i=-2}^{0}|\alpha^i| > 0$.

ii) ψ and ψ_1 are given by

$$\psi(s) = \alpha^0 dL_0[z_{t_0},z_{t^*-h,t^*},t^*;0,Y_{t^*-h,t^*}(s,\cdot),0]$$

$$- \alpha^{-1}\int_{-h}^{0} \zeta(\theta)Y(s,t^*+\theta)d\theta,$$

$$\psi_1(s,\theta) = 2\zeta(\theta)Y(s,t^*+\theta),$$

and satisfy

$$\psi(s) = 0 \quad \text{for} \quad s > t^*,$$

$$\psi_1(s,\theta) = 0 \quad \text{for} \quad s > t^*+\theta,$$

$$\psi(t^*) = \alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, Y_{t^*-h,t^*}(t^*,\cdot), 0],$$

$$\psi_1(s,\theta) = 2\zeta(\theta) \quad \text{for} \quad s = t^*+\theta,$$

$$\psi(s) = \alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, E^s_{t^*-h,t^*}, 0]$$

$$-\alpha^{-1}\int_{-h}^{0} \zeta(\theta)E^s(t^*+\theta)d\theta + \int_{s}^{t^{*+}} d_\alpha[\psi(\alpha)]\mu(\alpha,s)$$

$$-\int_{s}^{t^*} \psi(\alpha)\eta^*(\alpha,s)d\alpha \quad \text{for} \quad s \in [t_0,t^*),$$

where $E^s(t)$ is defined after (4.19),

$$\psi_1(s,\theta) = 2\zeta(\theta) + \int_{s}^{t^{*+}} d_\alpha[\psi_1(\alpha,\theta)]\mu(\alpha,s) - \int_{s}^{t^*} \psi_1(\alpha,\theta)\eta^*(\alpha,s)d\alpha$$

$$\text{for} \quad s \in [t_0,t^*+\theta),$$

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; 0, \dot{z}_{t^*-h,t^*}, 1] - \alpha^{-1}\int_{-h}^{0} \zeta(\theta)\dot{\zeta}(\theta)d\theta$$

$$\in \overline{\text{co}}\{-2\alpha^{-2}\zeta(\theta)\dot{\zeta}(\theta) \colon \theta \in [-h,0]\},$$

where $\overline{\text{co}}$ denotes closed convex hull,

$$\alpha^0 dL_0[z_{t_0}, z_{t^*-h,t^*}, t^*; Y_{t_0}(t_0,\cdot), 0, 0] + \psi(t_0)$$

$$\in \overline{\text{co}}[\{-\alpha^{-2}\psi_1(t_0,\theta) \colon \theta \in [-h,0]\} \cup \{-\alpha^{-2}\psi_1(t_0,\theta^-) \colon \theta \in [-h,0]\}].$$

iii) the following maximum conditions holds:

$$\int_{t_0}^{t^*} \psi(s)[f(z(\cdot),s) - f^*(z(\cdot),s)]ds$$

$$\leq |\alpha^{-2}| \sup_{\theta\in[-h,0]} \{\int_{t_0}^{t^*} \psi_1(s,\theta)[f(z(\cdot),s)-f^*(z(\cdot),s)]ds\}$$

for all $f \in \mathcal{F}$.

Remark 4.10. The proof follows the same pattern as the three given above and will not be presented here. The proof of the last condition in part ii) involves interchanging a limit and a supremum; it is quite tedious. If $\Phi = \{\varphi^*\}$, this condition need not hold, and the assumptions on μ and η^* in the statement of the theorem may be replaced by the assumption that $\mu(\cdot,\theta)$ is of bounded variation over bounded intervals for $\theta \in [\alpha_0,a)$.

Remark 4.11. Conditions for the map $s \to \int_s^t w(\alpha)\eta^*(\alpha,s)d\alpha$ to be continuous for all $w \in BV([\alpha_0,a),R^n)$ were given in remark 2.5. One obtains an η^* as specified there if the quasiconvex family has elements of the type

$$f(x(\cdot),t) = \tilde{f}(x(t),x(g_1(t)),x(g_2(t)),\ldots,x(g_q(t)),$$

$$\int_{\alpha_0}^{t} d_\theta[\hat{\eta}(t,\theta)]G(x(\theta),\theta),t),$$

where the g_ℓ are as in remark 2.5, $\hat{\eta}$ satisfies the same hypotheses as $\bar{\eta}$ in remark 2.5, and G is C^1 in x and Borel-measurable in t.

5. Sufficient Conditions and Existence

We present two types of problems for which the necessary conditions of Chapter 4 are also sufficient conditions. Suppose for fixed $T > t_0$ we are given the $n-1$ equations

$$(5.1) \qquad \frac{d}{dt}\Big[\bar{x}(t) - \int_{\alpha_0}^{t} d_\theta[\bar{\mu}(t,\theta)]\bar{x}(\theta)\Big] = \int_{\alpha_0}^{t} d_\theta[\bar{\eta}(t,\theta)]\bar{x}(\theta) + \bar{k}(u(\cdot),s)$$

$$\text{a.e. on } [t_0,T],$$

$$(5.2) \qquad \bar{x}_{t_0} = \bar{\varphi}, \quad \bar{\varphi} \text{ a fixed element of } C([\alpha_0,t_0],R^{n-1});$$

the set of restraints

$$(5.3) \qquad \bar{L}_i(\bar{x}_{T-h,T}) = 0, \quad i = 1,\dots,m,$$

$$(5.4) \qquad u \in \Omega = \{v: v \text{ measurable on } [t_0,T],\ v(s) \in U(s) \text{ for } s \in [t_0,T]\}$$

$$\text{where } \mathscr{U} \subset R^r,\ U: [t_0,T] \to \text{subsets of } \mathscr{U};$$

and the cost function

$$(5.5) \qquad J(u) = \bar{g}(\bar{x}_{T-h,T}) + \int_{t_0}^{T} [f^n(\bar{x}(s),s) + k^n(u(\cdot),s)]ds.$$

Let $\bar{\mu}$ satisfy (2.4), (2.5), and (3.1), $\bar{\mu}(\cdot,s)$ be of bounded variation on $[t_0,T]$ for all $s \in [\alpha_0,T]$; $\bar{\eta}$ satisfy (2.8); $\bar{k}$ and k^n measurable in (u,s), $|\bar{k}(\Omega,s)| + |k^n(\Omega,s)| \leq M(s)$ for $s \in [t_0,T]$, $M \in L^1([t_0,T],R)$; $\bar{L}_i$ be linear, $i = 1,\dots,m$, $\{\bar{L}_i: i = 1,\dots,m\}$ be linearly independent; $f^n(\bar{y},s)$ be C^1 and

convex in $\bar{y}$ for each $s \in [t_0,T]$, measurable in s for fixed $\bar{y} \in R^{n-1}$, and for compact $X \subset R^{n-1}$ there exists $m^n \in L^1([t_0,T],R)$ such that $|f^n(\bar{y},s)| \leq m^n(s)$, $|\frac{\partial f^n}{\partial \bar{x}}(\bar{y},s)| \leq m^n(s)$ for $\bar{y} \in X$, $s \in [t_0,T]$; $\bar{g}(\psi)$ be C^1 and convex in $\psi \in C([-h,0],R^{n-1})$.

We now augment the system, setting $x = (\bar{x},x^n)$, $x_{t_0} = (\bar{\varphi},0)$, $k(u(\cdot),s) = (\bar{k}(u(\cdot),s), k^n(u(\cdot),s))$, $f(x(\cdot),s) = (\int_{\alpha_0}^{s} d_\theta[\bar{\eta}(s,\theta)]\bar{x}(\theta), f^n(\bar{x}(s),s))$, and $g(x_{T-h,T}) = \bar{g}(\bar{x}_{T-h,T})$. Assume that $\mathcal{F} = \{F\colon F(x(\cdot),s) = f(x(\cdot),s) + k(u(\cdot),s),\ u \in \Omega\}$ is a quasi-convex family. (For example, satisfies the hypotheses of lemma 3.3. If k is linear in u, there may be delays in the control; see Banks and Jacobs [2]). The augmented problem is a special case of problem 4.2, where $L_i(x_{t_0},x_{T-h,T}) = \bar{L}_i(\bar{x}_{T-h,T})$, $i = 1,\ldots,m$, and $L_0(x_{t_0},x_{T-h,T}) = g(x_{T-h,T}) + x^n(T)$.

Theorem 5.1. Given the assumptions above and $u \in \Omega$ with response z satisfying the necessary conditions of theorem 4.2; then if $\alpha^0 \neq 0$, u is an optimal control.

Proof: Note that the assumptions of theorem 4.2 are satisfied. Let Y be the $n \times n$ matrix solution of (2.9)-(2.11) where

$$\mu(t,\theta) = \begin{pmatrix} \bar{\mu}(t,\theta) & 0 \\ 0 & 0 \end{pmatrix}, \quad \eta(t,\theta) = \begin{cases} 0 & \theta \geq t \\ \begin{pmatrix} \bar{\eta}(t,\theta) & 0 \\ -\frac{\partial f^n}{\partial \bar{x}}(\bar{z}(t),t) & 0 \end{pmatrix}, & \theta < t \end{cases},$$

and x be the response to a control $v \in \Omega$ such that $L_i(x_{T-h,T}) = 0$, $i = 1,\ldots,m$. Since $dg[z_{T-h,T};\cdot]$, $dL_i[z_{T-h,T};\cdot]$, $i = 1,\ldots,m$, are continuous linear functionals on $C([-h,0],R^n)$, there exist $\lambda_i : R \to R^{n*}$, $i = 0,\ldots,m$, of bounded variation, continuous from the left, $\lambda_i(s) = 0$, $s > 0$, and $\lambda_i(s) = \lambda_i(-h)$, $s \le -h$, $i = 0,\ldots,m$, such that for any $y \in C([-h,0],R^n)$,

$$dg[z_{T-h,T};y] = \int_{-h}^{0^+} d[\lambda_0(\theta)]y(\theta), \quad dL_i[z_{T-h,T};y] = \int_{-h}^{0^+} d[\lambda_i(\theta)]y(\theta),$$

$i = 1,\ldots,m$. Also, since neither g nor any of the L_i depend on z^n, $\lambda_i(s) = (\bar{\lambda}_i(s),0)$, $s \in R$, $i = 0,\ldots,m$. Hence, for ψ given by (4.16), from (4.19) we have

$$\psi(s) - \int_s^{T^+} d[\psi(\alpha)]\mu(\alpha,s) + \int_s^{T} \psi(\alpha)\eta(\alpha,s)d\alpha \tag{5.6}$$
$$= \alpha^0\{\int_s^{T^+} d_\theta[\lambda_0(\theta-T)] + (0,\ldots,0,1)\}$$
$$+ \Sigma_{i=1}^m \alpha^i \int_s^{T^+} d_\theta[\lambda_i(\theta-T)], \quad s \in [t_0,T].$$

By the form of the λ_i, μ, and η, for each $s \in [t_0,T]$, $\psi^n(s) = \alpha^0$.

Since x and z satisfy the augmented equations (5.1), (5.2),

$$0 = \int_{t_0}^{T^+} d[\psi(s)][x(s)-z(s)] - \int_{t_0}^{T^+} d[\psi(s)]\left\{\int_{t_0}^{T^+} d_\theta[\mu(s,\theta)][x(\theta)-z(\theta)]\right.$$
$$+ \int_{t_0}^{s}\int_{t_0}^{T^+} d_\theta[\begin{pmatrix} \bar{\eta}(\sigma,\theta) & 0 \\ 0 & 0 \end{pmatrix}][x(\theta)-z(\theta)]d\sigma$$
$$+ \int_{t_0}^{s} \begin{pmatrix} 0 \\ f^n(\bar{x}(\sigma),\sigma) - f^n(\bar{z}(\sigma),\sigma) \end{pmatrix} d\sigma$$

$$+ \int_{t_0}^{s} [k(v(\cdot),\sigma)-k(u(\cdot),\sigma)]d\sigma \Big\} .$$

$$0 = \int_{t_0}^{T^+} d[\psi(\theta)][x(\theta)-z(\theta)] - \int_{t_0}^{T^+} d[\psi(s)]\int_{t_0}^{T^+} d_\theta[\mu(s,\theta)][x(\theta)-z(\theta)]$$

$$- \int_{t_0}^{T^+} d[\psi(s)]\{\int_{t_0}^{s} \int_{t_0}^{T^+} d_\theta[\eta(\sigma,\theta)][x(\theta)-z(\theta)]d\sigma\}$$

$$- \int_{t_0}^{T^+} d[\psi(s)]\int_{t_0}^{s} \begin{pmatrix} 0 \\ f^n(\overline{x}(\sigma),\sigma)-f^n(\overline{z}(\sigma),\sigma)- \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(\sigma),\sigma)[\overline{x}(\sigma)-\overline{z}(\sigma)] \end{pmatrix} d\sigma$$

$$- \int_{t_0}^{T^+} d[\psi(s)]\int_{t_0}^{s} [k(v(\cdot),\sigma)-k(u(\cdot),\sigma)]d\sigma.$$

Integrating by parts and noting that $\psi(T^+) = 0$,

$$0 = \int_{t_0}^{T^+} d[\psi(\theta)][x(\theta)-z(\theta)] - \int_{t_0}^{T^+} d[\psi(s)]\int_{t_0}^{T^+} d_\theta[\mu(s,\theta)][x(\theta)-z(\theta)]$$

$$+ \int_{t_0}^{T} \psi(s)\int_{t_0}^{T^+} d_\theta[\eta(s,\theta)][x(\theta)-z(\theta)]ds$$

$$+ \int_{t_0}^{T} \psi(s) \begin{pmatrix} 0 \\ f^n(\overline{x}(s),s)-f^n(\overline{z}(s),s)- \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(s),s)[\overline{x}(s)-\overline{z}(s)] \end{pmatrix} ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s)-k(u(\cdot),s)]ds.$$

Changing some orders of integration and using $\psi^n(s) = \alpha^0$,

$$0 = \int_{t_0}^{T^+} d_\theta\{\psi(\theta)-\int_{t_0}^{T^+} d[\psi(s)]\mu(s,\theta) + \int_{t_0}^{T} \psi(s)\eta(s,\theta)ds\}[x(\theta)-z(\theta)]$$

$$+ \alpha^o \int_{t_0}^{T} \{f^n(\overline{x}(s),s)-f^n(\overline{z}(s),s) - \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(s),s)[\overline{x}(s)-\overline{z}(s)]\}ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s) - k(u(\cdot),s)]ds.$$

By (5.6),

$$0 = \alpha^o\{-\int_{t_0}^{T^+} d_\theta[\lambda_0(\theta-T)][x(\theta)-z(\theta)] - [x^n(T)-z^n(T)]\}$$

$$- \Sigma_{i=1}^{m}\alpha^i \int_{t_0}^{T^+} d_\theta[\lambda_i(\theta-T)][x(\theta)-z(\theta)]$$

$$+ \alpha^o \int_{t_0}^{T} \{f^n(\overline{x}(s),s)-f^n(\overline{z}(s),s)- \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(s),s)[\overline{x}(s)-\overline{z}(s)]\}ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s) - k(u(\cdot),s)]ds.$$

By the definition of the λ_i, linearity of the L_i, and the fact that $L_i(x_{T-h,T}) = L_i(z_{T-h,T})$, $i = 1,\ldots,m$,

$$0 = \alpha^o\{-dg[z_{T-h,T};x_{T-h,T}-z_{T-h,T}] - [x^n(T)-z^n(T)]\}$$

$$+ \alpha^o \int_{t_0}^{T} \{f^n(\overline{x}(s),s)-f^n(\overline{z}(s),s) - \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(s),s)[\overline{x}(s)-\overline{z}(s)]\}ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s) - k(u(\cdot),s)]ds.$$

Since $\alpha^o \neq 0$, we may assume $\alpha^o = -1$;

$$J(u)-J(v) = g(z_{T-h,T}) + z^n(T) - g(x_{T-h,T}) - x^n(T)$$

$$= -\{g(x_{T-h,T})-g(z_{T-h,T})\} + z^n(T) - x^n(T)$$

$$-\{-dg[z_{T-h,T};x_{T-h,T}-z_{T-h,T}]\} + x^n(T) - z^n(T)$$

$$- \int_{t_0}^{T} \{f^n(\overline{x}(s),s)-f^n(\overline{z}(s),s)- \frac{\partial f^n}{\partial \overline{x}}(\overline{z}(s),s)[\overline{x}(s)-\overline{z}(s)]\}ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s) - k(u(\cdot),s)]ds.$$

By the convexity of g, convexity of $f^n(\cdot,s)$ for each $s \in [t_0,T]$, and the assumption that u satisfies the maximum condition, $J(u)-J(v) \leq 0$. Thus u is an optimal control.

Remark 5.1. If we are dealing with the free endpoint problem, the necessary conditions ensure that $\alpha^0 \neq 0$, and so we need not assume it explicitly.

Now suppose we are given equations (5.1), (5.2); restraints

(5.7) $$\overline{\zeta}(\theta)[\overline{\zeta}(\theta) - \overline{x}(T+\theta)] \leq 0, \quad \theta \in [-h,0],$$

(5.8) $$[\overline{x}(T+\theta)]^2 - [\overline{\zeta}(\theta)]^2 \leq 0, \quad \theta \in [-h,0],$$

where $\overline{\zeta} \in C([-h,0],R^{n-1})$, $\overline{\zeta} \neq 0$, and (5.4); and cost function (5.5). Augmenting the system as before, we obtain a special case of problem 4.3, with $\zeta = (\overline{\zeta},0)$, $W = \begin{pmatrix} \overline{E} & 0 \\ 0 & 0 \end{pmatrix}$, and $L_0(x_{t_0},x_{T-h,T}) = g(x_{T-h,T}) + x^n(T)$. Assume the condition given before theorem 5.1.

Theorem 5.2. Given the assumptions above and $u \in \Omega$ with response z

satisfying the necessary conditions of theorem 4.3; then if $\alpha^0 \neq 0$, u is an optimal control.

Proof: Let Y be as in theorem 5.1, x the response to a control $v \in \Omega$ such that $Wx_{T-h,T} = \zeta = Wz_{T-h,T}$. Assume $\alpha^0 = -1$. Then for ψ given by (4.35), from (4.38) we have, as in the proof of theorem 5.1,

$$(5.9) \qquad \psi(s)-\int_s^{T^+} d[\psi(\alpha)]\mu(\alpha,s) + \int_s^T \psi(\alpha)\eta(\alpha,s)d\alpha$$

$$= \begin{cases} -[\int_{s-T}^{0^+} d_\theta\lambda_0(\theta) + (0,\ldots,0,1)] + \int_{s-T}^{0^+} \zeta(\theta)d\lambda(\theta), & T-h \le s \le T, \\ -[\int_{-h}^{0^+} d_\theta\lambda_0(\theta) + (0,\ldots,0,1)] + \int_{-h}^{0^+} \zeta(\theta)d\lambda(\theta), & t_0 \le s \le T-h. \end{cases}$$

Again, $\psi^n(s) = -1$. We also note that

$$\int_{T-h}^{T} d_s\{\int_{s-T}^{0^+} \zeta(\theta)d\lambda(\theta)\}[x(s)-z(s)]$$

$$= \int_{T-h}^{T} d_s\{\int_{s-T}^{0^+} \zeta(\theta)d\lambda(\theta)\}W[x(s)-z(s)] = 0.$$

Using this in place of $\sum_{i=1}^m \alpha^i L_i(x_{T-h,T}-z_{T-h,T}) = 0$, we obtain as in theorem 5.1,

$$0 = -\{-dg[z_{T-h,T};x_{T-h,T}-z_{T-h,T}] - [x^n(T) - z^n(T)]\}$$

$$- \int_{t_0}^{T} \{f^n(\bar{x}(s),s)-f^n(\bar{z}(s),s)- \frac{\partial f^n}{\partial \bar{x}}(\bar{z}(s),s)[\bar{x}(s)-\bar{z}(s)]\}ds$$

$$+ \int_{t_0}^{T} \psi(s)[k(v(\cdot),s) - k(u(\cdot),s)]ds.$$

$$\begin{aligned} J(u)-J(v) = & -\{g(x_{T-h,T})-g(z_{T-h,T})-dg[z_{T-h,T};x_{T-h,T}-z_{T-h,T}]\} \\ & -\int_{t_0}^{T}\{f^n(\bar{x}(s),s)-f^n(\bar{z}(s),s)-\frac{\partial f^n}{\partial \bar{x}}(\bar{z}(s),s)[\bar{x}(s)-\bar{z}(s)]\}ds \\ & +\int_{t_0}^{T}\psi(s)[k(v(\cdot),s)-k(u(\cdot),s)]ds\ . \end{aligned}$$

By the convexity of g, convexity of $f^n(\cdot,s)$ for each $s \in [t_0,T]$, and the assumption that u satisfies the maximum condition, $J(u) - J(v) \leq 0$. Thus u is an optimal control.

We now give conditions for an "attainable set" to be compact. Let (S,ρ) and (X,d) be metric spaces, $\mathscr{L}(X)$ be the collection of closed subsets of X. For a set $A \subset X$, define

$$J_\varepsilon[A] = \{x \in X\text{: there exists } y \in A \text{ such that } d(x,y) < \varepsilon\}.$$

Definition 5.1. The mapping $F\colon S \to \mathscr{L}(X)$ is said to be upper semicontinuous (abbreviated u.s.c.) at $t_0 \in S$ if $\limsup_{t \to t_0} F(t) \leq F(t_0)$, where supremums are taken in $\mathscr{L}(X)$ ordered by set inclusion.

Definition 5.2. The mapping $F\colon S \to \mathscr{L}(X)$ is said to be upper semicontinuous

with respect to inclusion (abbreviated u.s.c.i.) at $t_0 \in S$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ so that $\rho(t,t_0) < \delta$ implies $F(t) \subset J_\varepsilon[F(t_0)]$.

We consider the system of equations

(5.10) $$\frac{d}{dt}[D(x(\cdot),t)] = f(u,x(\cdot),t),$$

(5.11) $x_{t_0} = \varphi \in \Psi$, Ψ a compact set in $C([\alpha_0,t_0],R^n)$,

under the following assumptions:

(5.12) Conditions (2.3)-(2.5) and (3.1) hold.

(5.13) The mapping $U: [t_0,T] \to \mathscr{L}(R^r)\setminus\{\emptyset\}$ is u.s.c. on $[t_0,T]$, T a given constant, $T > t_0$.

(5.14) $f: M \times C([\alpha_0,T],R^n) \times [t_0,T] \to R^n$ is such that $(u,\psi,t) \to f(u,\psi(\cdot),t)$ is continuous in (u,ψ) for fixed $t \in [t_0,T]$ and measurable in t for fixed $(u,\psi) \in M \times C([\alpha_0,T],R^n)$, where $M = \bigcup_{t\in[t_0,T]} U(t)$.

(5.15) There is an integrable function $A: [t_0,T] \to R$, $A > 0$, such that if $(u,\psi_1,t), (u,\psi_2,t) \in M \times C([\alpha_0,T],R^n) \times [t_0,T]$, then $|f(u,\psi_1(\cdot),t)-f(u,\psi_2(\cdot),t)| \leq A(t)\|\psi_1-\psi_2\|_t$.

(5.16) For all $u \in U(t)$, $|f(u,\psi(\cdot),t)| \leq \nu(t)[B + \|\psi\|_t]$, where ν is

integrable on $[t_0,T]$ and B is a positive constant.

(5.17) π is a non-empty, closed subset of $[t_0,T]$.

(5.18) $\Omega(U,\pi) = \{(u,t_1): t_1 \in \pi,\ u: [t_0,t_1] \to R^r,\ u$ measurable, $u(t) \in U(t)$ a.e. on $[t_0,t_1]\}$.

(5.19) The set $f(U(t),\psi(\cdot),t)$ is a closed, convex subset of R^n for each fixed $(\psi,t) \in C([\alpha_0,T],R^n) \times [t_0,T]$.

(5.20) The set-valued mapping $\psi \in C([\alpha_0,t],R^n) \to f(U(t),\psi(\cdot),t)$ is u.s.c. for each fixed $t \in [t_0,T]$.

The conditions (5.12)-(5.18) are sufficient to guarantee that, corresponding to each $\varphi \in \Psi$ and $(u,t_1) \in \Omega(U,\pi)$, there is a unique continuous function (response), $x(\cdot;\varphi,u): [\alpha_0,t_1] \to R^n$, satisfying (5.10) a.e. on $[t_0,t_1]$ and satisfying (5.11). Let $h \geq 0$ be fixed.

Definition 5.3. A point $\zeta \in C([-h,0],R^n)$ is said to be attainable if there are a $\varphi \in \Psi$ and a $(u,t_1) \in \Omega(U,\pi)$ such that the corresponding response $x(\cdot;\varphi,u)$ satisfies $x_{t_1-h,t_1}(\cdot;\varphi,u) = \zeta$. The attainable set $\mathscr{R}$ is defined by $\mathscr{R} = \{\zeta \in C([-h,0],R^n): \zeta$ is attainable$\}$.

Theorem 5.3. If the system (5.10) with initial condition (5.11) satisfies (5.12)-(5.20), then $\mathscr{R}$ is compact.

Proof: We first show $\mathscr{R}$ is conditionally compact. Without loss of generality, assume $T \in \pi$. Let $(v^*,T) \in \Omega(U,\pi)$, and for any $(v,\tau) \in \Omega(U,\pi)$ define

$$v'(t) = \begin{cases} v(t) & \text{if } t_0 \le t \le \tau, \\ v^*(t) & \text{if } \tau < t \le T. \end{cases}$$

Then $(v',T) \in \Omega(U,\pi)$. From (5.10) and (5.16),

$$\left|\frac{d}{dt}D(x(\cdot;\varphi,v'),t)\right| \le \nu(t)[B + \|x(\cdot;\varphi,v')\|_t].$$

Choose $\gamma > 0$ such that $0 < \delta(\gamma) < 1$; let $b = [1-\delta(\gamma)]^{-1}$. Then, for $t \in [t_0,t_0+\gamma]$,

$$\|x(\cdot;\varphi,v')\|_t \le \|\varphi\| + 2\delta(T-\alpha_0)\|\varphi\| + \delta(\gamma)\|x(\cdot;\varphi,v')\|_t$$

$$+ \int_{t_0}^{t} \nu(s)[B + \|x(\cdot;\varphi,v')\|_s]ds,$$

$$\|x(\cdot;\varphi,v')\|_t \le b[1+2\delta(T-\alpha_0)]\|\varphi\| + b\int_{t_0}^{t} \nu(s)[B+\|x(\cdot;\varphi,v')\|_s]ds,$$

and so

$$B+\|x(\cdot;\varphi,v')\|_t \le b[1+2\delta(T-\alpha_0)][B+\|\varphi\|] + b\int_{t_0}^{t} \nu(s)[B+\|x(\cdot;\varphi,v')\|_s]ds.$$

By Gronwall's inequality, on $[t_0,t_0+\gamma]$,

$$B+\|x(\cdot;\varphi,v')\|_t \le b[1+2\delta(T-\alpha_0)][B+\|\varphi\|]\exp\{b\int_{t_0}^{t} \nu(s)ds\}.$$

In a similar manner on $[t_0+\gamma, t_0+2\gamma]$,

$$B+\|x(\cdot;\varphi,v')\|_t \leq b[1+2\delta(T-\alpha_0)][B+\|x(\cdot;\varphi,v')\|_{t_0+\gamma}]\exp\{b\int_{t_0+\gamma}^{t}\nu(s)ds\}$$

$$\leq b^2[1+2\delta(T-\alpha_0)]^2[B+\|\varphi\|]\exp\{b\int_{t_0}^{t}\nu(s)ds\}.$$

By a natural induction, if n_0 = 1 + greatest integer in $[\frac{T-t_0}{\gamma}]$,

$$(5.21)\quad B+\|x(\cdot;\varphi,v')\|_T \leq b^{n_0}[1+2\delta(T-\alpha_0)]^{n_0}[B+B_\Psi]\exp\{b\int_{t_0}^{T}\nu(s)ds\} = K,$$

where B_Ψ is a bound on the compact set Ψ. We define $\psi(\cdot;\varphi,v,\tau): [t_0,T] \to R^n$ by

$$(5.22)\quad \psi(t;\varphi,v,\tau) = f(v'(t),x(\cdot;\varphi,v'),t), \text{ for } \varphi \in \Psi,\ (v,\tau) \in \Omega(U,\pi).$$

From (5.16) and (5.21), for all $t \in [t_0,T]$, $\varphi \in \Psi$, and $(v,\tau) \in \Omega(U,\pi)$,

$$(5.23)\quad |\psi(t;\varphi,v,\tau)| \leq K\nu(t).$$

Since $f(v'(t),x(\cdot;\varphi,v'),t)$ is measurable in t, the $\psi(\cdot;\varphi,v,\tau)$ are measurable, and thus by (5.23) integrable over $[t_0,T]$. By (5.10), (5.11), (5.22), and (5.23), $x(\cdot;\varphi,v') \in C([\alpha_0,T],X)_{\Psi,K\nu}$, (see definition 3.1), where X is the compact ball in R^n of radius K. By lemma 3.1, $C([\alpha_0,T],X)_{\Psi,K\nu}$ is compact in $C([\alpha_0,T],R^n)$. Thus, by the Arzela-Ascoli theorem, the $x(\cdot;\varphi,v')$ are uniformly equicontinuous on $[\alpha_0,T]$. This implies that

$$\mathscr{R} = \{x_{\tau-h,\tau}(\cdot;\varphi,v'): \varphi \in \Psi,\ (v,\tau) \in \Omega(U,\pi)\}$$

is an equicontinuous subset of $C([-h,0],R^n)$, is bounded by K, and hence is

conditionally compact.

We now show $\mathscr{R}$ is closed. Let $\zeta_1, \zeta_2, \zeta_3, \ldots$ be a sequence of points in $\mathscr{R}$ such that $\zeta_k \to \zeta_0$ as $k \to \infty$. By the definition of $\mathscr{R}$, there exists a sequence $\varphi_k \in \Psi$, $k = 1,2,3,\ldots$, and a sequence $(u_k, t_k) \in \Omega(U,\pi)$, $k = 1,2,3,\ldots$, such that

$$(5.24) \qquad x_{t_k - h, t_k}(\cdot;\varphi_k, u_k) = \zeta_k \to \zeta_0 \quad \text{as} \quad k \to \infty.$$

Let $\psi_k = \psi(\cdot;\varphi_k, u_k, t_k)$. By (5.23), $\int_{t_0}^{T} |\psi_k(t)|\, dt \leq \int_{t_0}^{T} K\nu(t)\, dt$, so as elements of $L^1[t_0,T]$ the functions ψ_k form a bounded sequence. By (5.23) and [8; corollary IV.8.11] the set $\{\psi_k : k = 1,2,\ldots\}$ is weakly sequentially compact in $L^1[t_0,T]$. Hence there is a subsequence (still denoted by ψ_k) which converges weakly in $L^1[t_0,T]$ to an integrable function ψ_0, such that $\varphi_k \to \varphi^* \in \Psi$ as $k \to \infty$, and $t_k \to t^* \in \pi$ as $k \to \infty$. Thus, for each measurable $E \subset [t_0,T]$,

$$\lim_{k \to \infty} \int_E \psi_k(t)\, dt = \int_E \psi_0(t)\, dt.$$

Since the $x(\cdot;\varphi_k, u_k')$ are contained in the compact set $C([\alpha_0,T],X)_{\Psi,K\nu}$, there exists a further subsequence (still denoted $x(\cdot;\varphi_k,u_k')$) and a function $y \in C([\alpha_0,T],X)_{\Psi,K\nu}$ such that $\|x(\cdot;\varphi_k,u_k') - y\| \to 0$ as $k \to \infty$. Hence, uniformly in $t \in [t_0,T]$,

$$\lim_{k \to \infty} \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]x(\theta;\varphi_k,u_k') = \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]y(\theta).$$

Define the function $z: [\alpha_0,T] \to R^n$ by

$$z_{t_0} = \varphi^*,$$

$$z(t) = D(\varphi^*,t_0) + \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]y(\theta) + \int_{t_0}^{t} \psi_0(s)ds, \quad t_0 \le t \le T.$$

Using (5.10), (5.11), and (5.22) we have, for $t \in [t_0,T]$, $\lim_{k \to \infty} x(t;\varphi_k,u_k') = z(t)$. Thus $z = y$, and

$$\|x(\cdot;\varphi_k,u_k')-z\|_T \to 0 \quad \text{as} \quad k \to \infty, \tag{5.25}$$

$$z_{t_0} = \varphi^*, \tag{5.26}$$

$$z(t) = D(\varphi^*,t_0) + \int_{\alpha_0}^{t} d_\theta[\mu(t,\theta)]z(\theta) + \int_{t_0}^{t} \psi_0(s)ds, \quad t_0 \le t \le T.$$

We now show that, as $k \to \infty$, [see (5.24)]

$$x_{t_k-h,t_k}(\cdot;\varphi_k,u_k') = x_{t_k-h,t_k}(\cdot;\varphi_k,u_k) \to z_{t^*-h,t^*} = \zeta_0. \tag{5.27}$$

Define $\tau^-(t_k,s) = \min\{t_0,t_k+s\}$, $\tau^+(t_k,s) = \max\{t_0,t_k+s\}$. For the $s \in [-h,0]$ such that $t^*+s \le t_0$,

$$\begin{aligned}
|x(t_k+s;\varphi_k,u_k') - z(t^*+s)| &\le |x(t_k+s;\varphi_k,u_k') - \varphi_k(t^*+s)| \\
&\quad + |\varphi_k(t^*+s) - \varphi^*(t^*+s)| \\
&\le |x(\tau^+(t_k,s);\varphi_k,u_k') - \varphi_k(t_0)| \\
&\quad + |\varphi_k(\tau^-(t_k,s)) - \varphi_k(t^*+s)| \\
&\quad + |\varphi_k(t^*+s) - \varphi^*(t^*+s)|.
\end{aligned}$$

Since $t_k \to t^*$ as $k \to \infty$, $\tau^+(t_k, s) \to t_0$ uniformly in s as $k \to \infty$, so by the equicontinuity of $C([\alpha_0,T],X)_{\Psi,K\nu}$ the first term goes to zero uniformly in s as $k \to \infty$. By the uniform equicontinuity of Ψ, the second term goes to zero uniformly in s as $k \to \infty$. Since $\|\varphi_k - \varphi^*\| \to 0$ as $k \to \infty$, the third term goes to zero uniformly in s as $t \to \infty$. For the $s \in [-h,0]$ such that $t^*+s > t_0$,

$$\begin{aligned}
|x(t_k+s;\varphi_k,u_k') - z(t^*+s)| &\le |x(t_k+s;\varphi_k,u_k') - x(t^*+s;\varphi_k,u_k')| \\
&\quad + |x(t^*+s;\varphi_k,u_k') - z(t^*+s)| \\
&\le |\varphi_k(\tau^-(t_k,s)) - \varphi_k(t_0)| \\
&\quad + |g(x(\cdot;\varphi_k,u_k'),\tau^+(t_k,s)) - g(x(\cdot;\varphi_k,u_k'),t^*+s)| \\
&\quad + \left|\int_{t^*+s}^{\tau^+(t_k,s)} K\nu(\theta)\,d\theta\right| \\
&\quad + |x(t^*+s;\varphi_k,u_k') - z(t^*+s)|.
\end{aligned}$$

By the equicontinuity of Ψ and $t_k \to t^*$ as $k \to \infty$, the first term goes to zero uniformly in s as $k \to \infty$. By $C([\alpha_0,T],X)_{\Psi,K\nu} \times [t_0,T]$ compact and the joint continuity of g [see (2.4)], since $t_k \to t^*$ as $k \to \infty$ the second term goes to zero uniformly in s as $k \to \infty$. Since $\int K\nu(\theta)d\theta$ is absolutely continuous and $t_k \to t^*$ as $k \to \infty$, the third term goes to zero uniformly in s as $k \to \infty$. By (5.25) the last term goes to zero uniformly in s as $k \to \infty$. Thus we have shown that (5.27) holds.

The proof that $\mathscr{R}$ is closed will be complete if we show there is a u_0 such that $(u_0, t^*) \in \Omega(U,\pi)$ and

(5.28) $$x(t;\varphi^*,u_0) = z(t), \quad t_0 \le t \le t^*.$$

(Except for notation, the remainder of the proof is almost identical to that of Jacobs [18], theorem 4.1, following his equation (4.10). It is included here for completeness.) The subsequences $\{\psi_k,\ k \ge N\}$, $N = 1,2,3,\ldots,$ also converge weakly to ψ_0. Corresponding to each sequence $\{\psi_k,\ k \ge N\}$, there is a sequence $\{\psi^*_{kN}\}$ of convex linear combinations of the ψ_k, $k \ge N$, such that each $\{\psi^*_{kN}\}$ converges to ψ_0 in L^1-norm (strongly) [8 ; corollary V.3.14]. The sequences $\{\psi^*_{kN}\}$, $N = 1,2,3,\ldots,$ also converge to ψ_0 in measure [8; theorem III.3.6]. Thus, for each $N = 1,2,3,\ldots,$ there is a subsequence of $\{\psi^*_{kN}\}$ (still denoted $\{\psi^*_{kN}\}$) such that $\{\psi^*_{kN}\}$ converges pointwise to ψ_0 a.e. on $[t_0,T]$, $N = 1,2,3,\ldots$. Let E_N denote the subset of $[t_0,T]$ on which $\{\psi^*_{kN}\}$ does not converge pointwise to ψ_0, $N = 1,2,3,\ldots$. Then the set $E_0 = \bigcup_{N=1}^{\infty} E_N$ has Lebesgue measure zero. Let $t \in [t_0,T]\backslash E_0$ be fixed. Let B_0 be a ball in $C([\alpha_0,t],R^n)$ which is large enough that $x(\cdot;\varphi_k,u'_k) \in B_0$, $k = 1,2,3,\ldots$ [see (5.21)]. Then, by (5.16), (5.20), and [18; theorem 2.4], the mapping $x \in B_0 \to f(U(t),x(\cdot),t)$ is u.s.c.i. Thus, given $\varepsilon > 0$, there is a $\delta_\varepsilon > 0$ such that $x \in B_0$ and $\|x-z\|_t < \delta_\varepsilon$ imply $f(U(t),x(\cdot),t) \subset J_\varepsilon[f(U(t),z(\cdot),t)]$. By (5.25) there is a positive integer N_{δ_ε} such that $k \ge N_{\delta_\varepsilon}$ implies $\|x(\cdot;\varphi_k,u'_k)-z\|_t < \delta_\varepsilon$. Thus, when $k \ge N_{\delta_\varepsilon}$, $f(U(t),x(\cdot;\varphi_k,u'_k),t) \subset J_\varepsilon[f(U(t),z(\cdot),t)]$. Since $\psi_k(t) \in f(U(t),x(\cdot;\varphi_k,u'_k),t)$, $k = 1,2,3,\ldots,$ and since $J_\varepsilon[f(U(t),z(\cdot),t)]$ is convex [see (5.19)], it follows that $\{\psi^*_{kN_{\delta_\varepsilon}}(t)\} \subset J_\varepsilon[f(U(t),z(\cdot),t)]$. Consequently, $\psi_0(t)$ belongs to the closure of $J_\varepsilon[f(U(t),z(\cdot),t)]$. Since $\varepsilon > 0$ was arbitrary, we must have $\psi_0(t) \in f(U(t),z(\cdot),t)$. Thus $\psi_0(s) \in f(U(s),z(\cdot),s)$ a.e. on $[t_0,T]$. By (5.13) and [18; theorem 3.1], there is a

measurable function $u_0 : [t_0, t^*] \to R^r$ such that $u_0(t) \in U(t)$, $t_0 \leq t \leq t^*$, and $\psi_0(t) = f(u_0(t), z(\cdot), t)$ a.e. on $[t_0, t^*]$. Using (5.26), we have that (5.28) holds.

Since we have shown $\mathscr{R}$ to be conditionally compact and closed, the proof is complete.

Let (S, ρ) be a locally compact metric space, (X, d) a metric space. The following more general statement of [18; theorem 2.2] is also a special case of results in [17 ; chapter 2].

Lemma 5.1. Given a mapping $F : S \to \mathscr{L}(X)$, the following are equivalent:

i) F is u.s.c. at $t_0 \in S$;

ii) if $\{t_n\}$, $\{P_n\}$ are sequences in S and X respectively such that $P_n \in F(t_n)$, $n = 1, 2, 3, \ldots$, and $t_n \to t_0$, $P_n \to P_0$ as $n \to \infty$, then $P_0 \in F(t_0)$.

Let there be given a mapping $\mathscr{T} : [t_0, T] \to \mathscr{L}(C(-h, 0], R^n)) \setminus \{\emptyset\}$ which is u.s.c. on $[t_0, T]$. Define

$$\hat{\mathscr{R}} = \{x_{t_1 - h, t_1}(\cdot; \varphi, u) \in \mathscr{R} : x_{t_1 - h, t_1}(\cdot; \varphi, u) \in \mathscr{T}(t_1)\}.$$

Theorem 5.4. Given (5.10)-(5.20), $\mathscr{T}$ as above, and a continuous function $p : C([-h, 0], R^n) \to R$; then if $\hat{\mathscr{R}}$ is nonempty there exists $(\varphi^*, u^*, t^*) \in \Psi \times \Omega(U, \pi)$ such that $x_{t^* - h, t^*}(\cdot; \varphi^*, u^*) \in \hat{\mathscr{R}}$ and

$$p(x_{t^* - h, t^*}(\cdot; \varphi^*, u^*)) = \min\{p(\zeta) : \zeta \in \hat{\mathscr{R}}\}$$

Proof: By theorem 5.3 and lemma 5.1, $\hat{\mathscr{R}}$ is closed, and hence by theorem 5.3 is compact. This and p continuous imply the result.

6. Partial Differential Equations and Examples

The theory above may be applied to certain linear hyperbolic partial differential equations with boundary controls. For this discussion, subscript t or x denotes the respective partial derivative, and prime denotes the total derivative. Suppose we are given the equation

$$(6.1)\qquad w_{tt}(t,x) - c^2 w_{xx}(t,x) = 0, \quad t \in [0,\infty), \quad x \in [0,1],$$

with initial conditions

$$(6.2)\qquad w(0,x) = f_0(x), \quad w_t(0,x) = f_1(x) \quad \text{on} \quad [0,1],$$

and the boundary conditions

$$(6.3)\qquad \begin{aligned} A_0(t)w_{tt}(t,0) + B_0(t)w_{tx}(t,0) &= G_0(t,w(t,0),w_t(t,0),w_x(t,0)), \\ A_1(t)w_{tt}(t,1) + B_1(t)w_{tx}(t,1) &= G_1(t,w(t,1),w_t(w,1),w_x(t,1)), \end{aligned}$$

where the initial functions satisfy the boundary conditions at $t = 0$.

This is a controlled system if

$$G_0(t,\rho,\sigma,\tau) = F_0(t,u(t),\rho,\sigma,\tau),$$

$$G_1(t,\rho,\sigma,\tau) = F_1(t,v(t),\rho,\sigma,\tau),$$

and $\binom{u}{v}$ is the control. We make the following assumptions:

(6.4) $f_0'(x)$ and $f_1(x)$ are absolutely continuous, with derivatives in $L^2[0,1]$.

(6.5) The A_i and B_i are continuous, the G_i are measurable in t and continuous in (w, w_t, w_x), and for each compact $Z \subset R^3$ there exists $K_i \in L^2_{loc}([0,\infty),R)$ such that $|G_i(s,y)| \leq K_i(s)$ for every $s \in [0,\infty)$ and $y \in Z$, $i = 0,1$.

(6.6) For each $s \in [0,\infty)$,

$$[A_0(s) - \frac{1}{c} B_0(s)] \neq 0, \quad [A_1(s) + \frac{1}{c}B_1(s)] \neq 0.$$

Motivated by partial differential equations and the paper by Cooke and Krumme [7], we assume a solution of the form

$$w(t,x) = \varphi(t + \frac{x}{c}) + \psi(t - \frac{x}{c}). \tag{6.7}$$

Substituting this into (6.3), and setting

$$\alpha_i(s) = [A_i(s) + \frac{1}{c}B_i(s)], \ \beta_i(s) = [A_i(s) - \frac{1}{c}B_i(s)], \ i = 0,1,$$

we obtain

$$\begin{aligned}\alpha_0(t)\varphi''(t) + \beta_0(t)\psi''(t) &= G_0(t, f_0(0) + \int_0^t \varphi'(s)ds + \int_0^t \psi'(s)ds, \\ &\qquad \varphi'(t) + \psi'(t), \frac{1}{c}\varphi'(t) - \frac{1}{c}\psi'(t)) \\ &= \tilde{G}_0(\varphi'(\cdot), \psi'(\cdot), t), \quad 0 \leq t,\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}&\alpha_1(t - \frac{1}{c})\varphi''(t) + \beta_1(t - \frac{1}{c})\psi''(t - \frac{2}{c}) \\ &\qquad = G_1(t - \frac{1}{c}, f_0(0) + \int_0^t \varphi'(s)ds + \int_0^{t-\frac{2}{c}} \psi'(s)ds, \\ &\qquad\qquad \varphi'(t) + \psi'(t - \frac{2}{c}), \frac{1}{c}\varphi'(t) - \frac{1}{c}\psi'(t - \frac{2}{c}))\end{aligned}$$

$$= \tilde{G}_1(\varphi'(\cdot),\psi'(\cdot),t), \quad \frac{1}{c} \le t.$$

By (6.6) it is proper to multiply this pair of equations by the matrix

$$\begin{pmatrix} 0 & \dfrac{1}{\alpha_1(t-\frac{1}{c})} \\ \dfrac{1}{\beta_0(t)} & -\dfrac{\alpha_0(t)}{\alpha_1(t-\frac{1}{c})\beta_0(t)} \end{pmatrix};$$

we then obtain

$$(6.9) \qquad \begin{pmatrix} \varphi''(t) \\ \psi''(t) \end{pmatrix} + \begin{pmatrix} \dfrac{\beta_1(t-\frac{1}{c})}{\alpha_1(t-\frac{1}{c})}\,\psi''(t-\frac{2}{c}) \\ -\dfrac{\alpha_0(t)\beta_1(t-\frac{1}{c})}{\alpha_1(t-\frac{1}{c})\beta_0(t)}\,\psi''(t-\frac{2}{c}) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\alpha_1(t-\frac{1}{c})}\,\tilde{G}_1(\varphi'(\cdot),\psi'(\cdot),t) \\ \dfrac{1}{\beta_0(t)}\,\tilde{G}_0(\varphi'(\cdot),\psi'(\cdot),t) - \dfrac{\alpha_0(t)}{\alpha_1(t-\frac{1}{c})\beta_0(t)}\,G_1(\varphi'(\cdot),\psi'(\cdot),t) \end{pmatrix}$$

$$\text{for} \quad t \ge \frac{1}{c}.$$

From (6.2) and (6.7), for $x \in [0,1]$,

$$\frac{1}{c}\varphi'(\frac{x}{c}) - \frac{1}{c}\psi'(-\frac{x}{c}) = f_0'(x),$$

$$\varphi'(\frac{x}{c}) + \psi'(-\frac{x}{c}) = f_1(x).$$

Thus, for $x \in [0,1]$,

$$\varphi'(\tfrac{x}{c}) = \tfrac{c}{2}f_0'(x) + \tfrac{1}{2}f_1(x),$$

$$\psi'(-\tfrac{x}{c}) = -\tfrac{c}{2}f_0'(x) + \tfrac{1}{2}f_1(x).$$

(6.4) implies that φ' on $[0, \frac{1}{c}]$ and ψ' on $[-\frac{1}{c}, 0]$ are determined as absolutely continuous functions with L^2 derivatives. We use (6.8) to determine ψ' on $[0, \frac{1}{c}]$; by (6.5), (6.6), and (6.8), ψ' is absolutely continuous on $[0, \frac{1}{c}]$ with an L^2 derivative. Thus there is sufficient initial data to solve (6.9), with $\alpha_0 = -\frac{1}{c}$, $t_0 = \frac{1}{c}$; theorem 2.1 guarantees the existence of a continuous solution (φ', ψ'). From (6.5) and (6.9) we see that the solution is absolutely continuous, with a derivative in $L^2_{loc}([0,\infty), R^2)$. By construction, $w(t,x)$ given by (6.7) satisfies (6.2) and (6.3). The derivatives of $w(t,x)$ through second order obtained by formally differentiating (6.7) are derivatives of $w(t,x)$ in the ordinary and generalized sense (the latter from an integration by parts). Thus w may be said to solve (6.1) in either the generalized sense or in the sense of almost everywhere (which does not seem to be well-established in partial differential equations).

If we are given the terminal conditions

$$w(T,x) = g_0(x), \quad w_t(T,x) = g_1(x) \quad \text{on } [0,1],$$

where g_0, g_1 satisfy (6.4), we obtain, for $x \in [0,1]$,

$$\varphi'(T + \tfrac{x}{c}) = \tfrac{c}{2}\, g_0'(x) + \tfrac{1}{2}g_1(x),$$

$$\psi'(T - \tfrac{x}{c}) = -\tfrac{c}{2}g_0'(x) + \tfrac{1}{2}g_1(x),$$

exactly as for the initial conditions. These give terminal conditions as a target function for ψ' on $[T - \frac{1}{c}, T]$ and for φ' on $[T, T + \frac{1}{c}]$.

Remark 6.1. The form of the equation and terminal conditions above shows that it is reasonable to require that terminal conditions be given on $[\bar{T}-h, \bar{T}]$ even when the hereditary dependence of the neutral equation is truly of the $(x(\cdot),t)$ form.

Remark 6.2. Instead of the equation (6.1), suppose we have the coupled equations

$$Li_t = -v_x,$$

$$Cv_t = -i_x,$$

with initial conditions

$$i(0,x) = \hat{i}(x),\ v(0,x) = \hat{v}(x),\ x \in [0,1],$$

and boundary conditions

$$A_0(t)i_t(t,0) + B_0(t)v_t(t,0) = g_0(t,i(t,0),v(t,0)),$$

$$A_1(t)i_t(t,1) + B_1(t)v_t(t,1) = g_1(t,i(t,1),v(t,1)).$$

Let $\hat{i}$ and $\hat{v}$ be absolutely continuous, with derivatives in $L^2[0,1]$; the A_i, B_i and g_i satisfy conditions similar to (6.5) and (6.6).

Assume a solution of the form

$$v(t,x) = \frac{1}{2\sqrt{C}}[\psi(t - x\sqrt{LC}) - \varphi(t + x\sqrt{LC})],$$

$$i(t,x) = \frac{1}{2\sqrt{L}}[\psi(t - x\sqrt{LC}) + \varphi(t + x\sqrt{LC})].$$

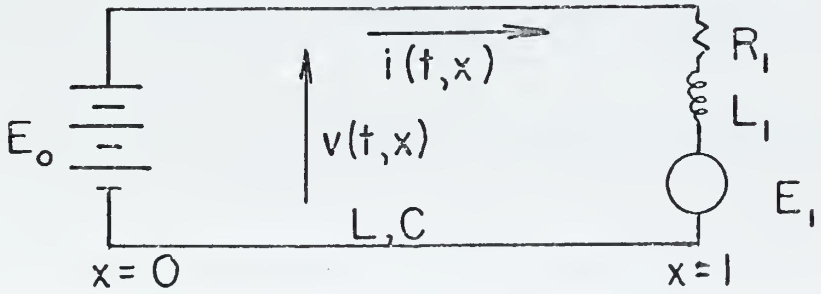
As above, we obtain an equation of neutral type in ψ and φ, whose solution will be absolutely continuous with locally L^2 derivative. This yields a pair (i,v), with $(i_t, i_x, v_t, v_x) \in L^2_{loc}([0,\infty) \times [0,1], R^4)$, which satisfies the coupled equations in the generalized sense.

Remark 6.3. It is not always necessary to have conditions (6.6) hold. For example, assume

$$A_0(s) \equiv B_0(s) \equiv 0, \quad [A_1(s) + \frac{1}{c}B_1(s)] \neq 0 \quad \text{for all} \quad s \geq 0.$$

If the boundary condition at $x = 0$ can be used to solve for ψ' as a function of φ', we substitute this into the boundary condition at $x = 1$ and obtain an equation of neutral type in φ' alone. The equation may then be converted to an equation in either w_t or w_x. This procedure, in connection with the coupled equation mentioned in remark 6.2, was used by Brayton [5, section 2].

Example 6.1. Consider the transmission line given by



where $E_0 = 200\text{v}$, $R_1 = 400\Omega$, $L_1 = 1\text{h}$, $|E_1| \le 75\text{v}$, $C = 10^{-6}\text{f}$, and $L = 0.16\text{h}$ (these last two values correspond to a transmission line roughly 10^5 meters long). The equations are

$$0.16 i_t(t,x) = -v_x(t,x),$$

$$10^{-6} v_t(t,x) = -i_x(t,x),$$

with boundary conditions

$$0 = 200 - v(t,0),$$

$$i_t(t,1) = v(t,1) - 400 i(t,1) - E_1(t).$$

If we assume a solution as in remark 6.2, we obtain the equations

$$\psi(t) = \varphi(t) + 0.4, \quad t \ge 0,$$

$$\psi'(t) + \psi'(t-h) = -800\psi(t) - 0.8E_1(t - \tfrac{h}{2}) + 320, \; t \ge \tfrac{h}{2}, \tag{6.10}$$

where $h = 2\sqrt{LC} = 8 \times 10^{-4}$ sec. By a different sequence of substitutions (similar to those detailed in Slemrod [25]) we obtain the equation

$$\frac{d}{dt}[i(t,1) + i(t-h,1)] = -800 i(t,1) - E_1(t) - E_1(t-h) + 400, \; t \ge h.$$

If the control is the value of E_1, these are equations of neutral type with delays in the control.

Suppose we are given initial conditions $i(0,x) \equiv 0$, $v(0,x) \equiv 200$;

then on $[-\frac{h}{2},0]$, $\psi(s) \equiv 0.2$, on $[0,\frac{h}{2}]$, $\varphi(s) \equiv -0.2$. Thus we may take (6.10) as the equation for $t \geq \frac{h}{2}$, with initial function $\psi(s) = 0.2$, $s \in [-\frac{h}{2}, \frac{h}{2}]$. To obtain an initial function for the equation in i, one must use another equation such as (6.10), and the initial function $i(s,1)$, $s \in [0,h]$, will depend on values of $E_1(s)$ for $s \in [0,h]$.

Attempts were made to determine the values of $E_1(t)$ (acting as a control variable) which would drive $i_{t^*-h,t^*}(\cdot,1)$ to the constant function $\zeta(\theta) = 0.4$, $\theta \in [-h,0]$, in minimum time t^*. The problem of hitting $i(t^*,1) = 0.4$ in minimum time was solved, but the trajectory could not be held at $i(s,1) = 0.4$ for $s \in (t^*,t^*+h]$. Despite repeated efforts, a solution was not obtained for the problem with a function target.

We now consider a less complicated version of problem 4.3, and the corresponding statement of theorem 4.3. Assume the equation has the form

$$\dot{x}(t) + A\dot{x}(t-h) = Bx(t) + Cx(t-h) + k(u(\cdot),t)$$

in R^{n-1}, with fixed initial function x_0. We wish to drive to the piecewise C^1 function $\zeta: [-h,0] \to R^{n-1}$, $\zeta \not\equiv 0$, in a manner such that the function $J(u) = \int_0^T f(x(s),u(s),s)ds$ is minimized over a given class of controls, Ω. Assume that $\{f(x(t),u(t),t): u \in \Omega\}$ and $\{k(u(\cdot),t): u \in \Omega\}$ are quasiconvex families of functions (for example, satisfy the conditions of lemma 3.3).

Theorem 6.1. Given the assumptions above, let u^* be an optimal control with response z. Then there exist a row n-vector valued function ψ defined on $[0,\infty)$, a real-valued function $\lambda = 2\lambda_2-\lambda_1$ defined on R, and a real number α^0, such that

i) λ_1 is a non-increasing function of bounded variation, continuous from the left, $\lambda_i = 0$ on $(0,\infty)$, λ_i constant on $(-\infty,-h]$,

$$i = 1,2;\ \alpha^0 \le 0;\ |\alpha^0| + \operatorname*{var}_{[-h,\infty)} \lambda_1 + \operatorname*{var}_{[-h,\infty)} \lambda_2 > 0.$$

ii) ψ is given by

$$\psi(s) = 0, \quad s > T,$$

$$\psi(T) = (-\zeta(0)\lambda(0), \alpha^0),$$

$$\psi(s) = \left(\int_{s-T}^{0^+} \zeta(\theta)d\lambda(\theta), \alpha^0\right) - \psi(s+h)\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ \int_{s+h}^{T} \psi(\alpha)\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} d\alpha + \int_{s}^{T} \psi(\alpha)\begin{pmatrix} B & 0 \\ \frac{\partial f}{\partial x}(z(\alpha), u^*(\alpha), \alpha) & 0 \end{pmatrix} d\alpha,$$

$$s \in [0,T).$$

iii) the following maximum condition holds:

$$\int_0^T \psi(s)\begin{pmatrix} k(u(\cdot),s) \\ f(z(s),u(s),s) \end{pmatrix} ds \le \int_0^T \psi(s)\begin{pmatrix} k(u^*(\cdot),s) \\ f(z(s),u^*(s),s) \end{pmatrix} ds$$

for all $u \in \Omega$.

Theorem 6.1 is applicable to each of the following examples; graphs of the corresponding optimal trajectories and optimal controls are at the end of the chapter.

Example 6.2. Consider the equation

$$\dot{x}(t) = \dot{x}(t-1) + u(t), \quad t \in [0,T];$$

here

$$\bar{\mu}(t,\theta) = \begin{cases} 0, & t-1 < \theta, \\ -1, & \theta \leq t-1, \end{cases} \qquad \bar{\eta}(t,\theta) \equiv 0.$$

Let $T = 3$, initial function $\bar{\varphi}(\theta) = 1$, $\theta \in [-1,0]$, target function $\bar{\zeta}(\theta) = 1-\theta$, $\theta \in [-1,0]$, and cost function $J(u) = \int_0^T u^2(t)dt$. Augment the system as in remark 4.4; the problem is now in the form of problem 4.3. The adjoint matrix is given by $Y_{12}(s,t) \equiv Y_{21}(s,t) \equiv 0$,

$$Y_{11}(s,t) = \begin{cases} 0, & t < s, \\ K, & t-K < s \leq t+1-K, \quad k = 1,2,3,\ldots, \end{cases}$$

$$Y_{22}(s,t) = \begin{cases} 0, & t < s, \\ 1, & s \leq t. \end{cases}$$

An optimal trajectory and control are

$$\bar{Z}(t) = \begin{cases} 1 + \frac{2}{3}t, & t \in [0,1], \\ \frac{4}{3} + \frac{1}{3}t, & t \in [1,2], \\ 4-t \quad , & t \in [2,3], \end{cases} \qquad u^*(t) = \begin{cases} \frac{2}{3}, & t \in (0,1), \\ -\frac{1}{3}, & t \in (1,2), \\ -\frac{4}{3}, & t \in (2,3). \end{cases}$$

$$\text{Let} \quad \alpha^0 = -1, \quad \lambda(\theta) = \begin{cases} \frac{1}{3}, & \theta \in (-\infty,-1], \\ \frac{8}{3}, & \theta \in (-1,0], \\ 0, & \theta \in (0,\infty); \end{cases}$$

$$\text{then} \quad \psi(s) = \begin{cases} (\frac{4}{3}, -1), & s \in (0,1), \\ (-\frac{2}{3}, -1), & s \in (1,2), \\ (-\frac{8}{3}, -1), & s \in (2,3). \end{cases}$$

The integral maximum condition is equivalent to the pointwise condition

$$\psi(s)\begin{pmatrix} v \\ v^2 \end{pmatrix} \leq \psi(s)\begin{pmatrix} u^*(s) \\ u^{*2}(s) \end{pmatrix} \quad \text{a.e. on} \quad (0,3),$$

for all $v \in R$. This implies $2u^*(s) = \psi^1(s)$. Theorem 5.2 guarantees that the solution given is optimal.

Example 6.3. Consider the same problem as example 6.2, with the added restriction that $|u(s)| \leq \frac{5}{4}$, $s \in [0,3]$. Then an optimal trajectory and control are

$$\bar{Z}(t) = \begin{cases} 1 + \frac{3}{4}t, & t \in [0,1], \\ \frac{3}{2} + \frac{1}{4}t, & t \in [1,2], \\ 4-t, & t \in [2,3], \end{cases} \qquad u^*(t) = \begin{cases} \frac{3}{4}, & t \in (0,1), \\ -\frac{1}{2}, & t \in (1,2), \\ -\frac{5}{4}, & t \in (2,3). \end{cases}$$

$$\text{Let} \quad \alpha^0 = -1, \quad \lambda(\theta) = \begin{cases} \frac{1}{2}, & \theta \in (-\infty,-1], \\ \frac{7}{2}, & \theta \in (-1,0], \\ 0, & \theta \in (0,\infty); \end{cases}$$

$$\text{then} \qquad \psi(s) = \begin{cases} (\frac{3}{2}, -1), & s \in (0,1), \\ (-1,-1), & s \in (1,2), \\ (-\frac{7}{2},-1), & s \in (2,3). \end{cases}$$

The maximum condition is as before, except that only $v \in [-\frac{5}{4}, \frac{5}{4}]$ are considered.

Example 6.4. Consider the same problem as example 6.2, with $T = \frac{5}{2}$. Then an optimal trajectory and control are

$$\bar{Z}(t) = \begin{cases} 1 + \frac{16}{15}t, & t \in [0, \frac{1}{2}], \\ \frac{43}{30} + \frac{1}{5}t, & t \in [\frac{1}{2}, 1], \\ \frac{27}{30} + \frac{11}{15}t, & t \in [1, \frac{3}{2}], \\ \frac{7}{2} - t, & t \in [\frac{3}{2}, \frac{5}{2}], \end{cases} \qquad u^*(t) = \begin{cases} \frac{16}{15}, & t \in (0, \frac{1}{2}), \\ \frac{1}{5}, & t \in (\frac{1}{2}, 1), \\ -\frac{1}{3}, & t \in (1, \frac{3}{2}), \\ -\frac{6}{5}, & t \in (\frac{3}{2}, 2), \\ -\frac{26}{15}, & t \in (2, \frac{5}{2}). \end{cases}$$

$$\text{Let} \quad \alpha^{\circ} = -1, \qquad \lambda(\theta) = \begin{cases} \frac{7}{45}, & \theta \in (-\infty, -1], \\ \frac{124}{45}, & \theta \in (-1, -\frac{1}{2}], \\ \frac{52}{15}, & \theta \in (-\frac{1}{2}, 0], \\ 0, & \theta \in (0, \infty); \end{cases}$$

$$\text{then} \quad \psi(s) = \begin{cases} (\frac{32}{15}, -1), & s \in (0, \frac{1}{2}), \\ (\frac{2}{5}, -1), & s \in (\frac{1}{2}, 1), \\ (-\frac{2}{3}, -1), & s \in (1, \frac{3}{2}), \\ (-\frac{12}{5}, -1), & s \in (\frac{3}{2}, 2), \\ (-\frac{52}{15}, -1), & s \in (2, \frac{5}{2}). \end{cases}$$

The maximum condition is as in example 6.2.

Example 6.5. Consider the same problem as example 6.2, except that

$$\bar{\zeta}(\theta) = \begin{cases} \frac{1}{2} - \theta, & \theta \in [-1, -\frac{1}{2}], \\ \frac{3}{2} + \theta, & \theta \in [-\frac{1}{2}, 0]. \end{cases}$$

Then an optimal trajectory and control are

$$\bar{Z}(t) = \begin{cases} 1 - \frac{1}{12}t, & t \in [0, \frac{1}{2}], \\ \frac{2}{3} + \frac{7}{12}t, & t \in [\frac{1}{2}, 1], \\ \frac{5}{3} - \frac{5}{12}t, & t \in [1, \frac{3}{2}] \\ -\frac{1}{3} + \frac{11}{12}t, & t \in [\frac{3}{2}, 2], \\ \frac{7}{2} - t, & t \in [2, \frac{5}{2}], \\ -\frac{3}{2} + t, & t \in [\frac{5}{2}, 3], \end{cases} \qquad u^*(t) = \begin{cases} -\frac{1}{12}, & t \in (0, \frac{1}{2}), \\ \frac{7}{12}, & t \in (\frac{1}{2}, 1), \\ -\frac{1}{3}, & t \in (1, \frac{3}{2}), \\ \frac{1}{3}, & t \in (\frac{3}{2}, 2), \\ -\frac{7}{12}, & t \in (2, \frac{5}{2}), \\ \frac{1}{12}, & t \in (\frac{5}{2}, 3). \end{cases}$$

Let $\alpha^{\circ} = -1$, $\lambda(\theta) = \begin{cases} \frac{1}{9}, & \theta \in (-\infty, -1], \\ \frac{11}{9}, & \theta \in (-1, -\frac{1}{2}], \\ -\frac{1}{9}, & \theta \in (-\frac{1}{2}, 0], \\ 0, & \theta \in (0, \infty); \end{cases}$

then $\psi(s) = \begin{cases} (-\frac{1}{6}, -1), & s \in (0, \frac{1}{2}), \\ (\frac{7}{6}, -1), & s \in (\frac{1}{2}, 1), \\ (-\frac{4}{6}, -1), & s \in (1, \frac{3}{2}), \\ (\frac{4}{6}, -1), & s \in (\frac{3}{2}, 2), \\ (-\frac{7}{6}, -1), & s \in (2, \frac{3}{2}), \\ (\frac{1}{6}, -1), & s \in (\frac{5}{2}, 3). \end{cases}$

The maximum condition is as in example 6.2.

Example 6.6. Consider the equation

$$\dot{\overline{x}}(t) = \overline{x}(t) - \overline{x}(t-1) + u(t), \quad t \in [0,T];$$

here

$$\overline{\mu}(t,\theta) \equiv 0, \quad \overline{\eta}(t,\theta) = \begin{cases} 0, & t \leq \theta, \\ -1, & t-1 < \theta < t, \\ 0, & \theta \leq t-1. \end{cases}$$

Let $T = 2$, initial function $\overline{\varphi}(\theta) = 0$, $\theta \in [-1,0]$, target function $\overline{\zeta}(\theta) = 2+\theta$, $\theta \in [-1,0]$, and cost function $J(u) = \int_0^T u^2(t)dt$. Augment the system as in remark 4.4. The adjoint matrix is given by $Y_{12}(s,t) \equiv Y_{21}(s,t) \equiv 0$,

$$Y_{11}(s,t) = \begin{cases} 0 & , \quad t < s, \\ e^{(t-s)} & , \quad t-1 < s \leq t, \\ e^{(t-s)} - (t-1-s)e^{(t-1-s)} & , \quad t-2 < s \leq t-1, \end{cases}$$

$$Y_{22}(s,t) = \begin{cases} 0 , & t < s, \\ 1 , & s \leq t. \end{cases}$$

An optimal trajectory and control are

$$\overline{Z}(t) = \begin{cases} \dfrac{\sinh(\sqrt{2}\,t)}{2\sinh\sqrt{2}} + \dfrac{t}{2} , & t \in [0,1], \\ t , & t \in [1,2]. \end{cases}$$

$$u^*(t) = \begin{cases} \dfrac{1}{2\sinh\sqrt{2}}[\sqrt{2}\cosh(\sqrt{2}\,t) - \sinh(\sqrt{2}\,t)] + \dfrac{(1-t)}{2}, & t \in (0,1), \\[2ex] \dfrac{\sinh[\sqrt{2}\,(t-1)]}{2\sinh\sqrt{2}} + \dfrac{(1-t)}{2}, & t \in (1,2). \end{cases}$$

Let $\alpha^0 = -1$, $te^t d\lambda(t-2) = d\beta(t)$, where

$$\beta(t) = \begin{cases} [1-\sqrt{2}\coth\sqrt{2}]e^1, & t \in (-\infty,1], \\[2ex] -\dfrac{1}{\sinh\sqrt{2}}[\exp\{(\sqrt{2}+1)t-\sqrt{2}\} - \exp\{-(\sqrt{2}-1)t+\sqrt{2}\}] + (t-1)e^t, & t\in(1,2], \\[2ex] 0, & t \in (2,\infty); \end{cases}$$

then

$$\psi(s) = \begin{cases} \left(\dfrac{1}{\sinh\sqrt{2}}[\sqrt{2}\cosh(\sqrt{2}s) - \sinh(\sqrt{2}s)] + 1-s, -1\right), & s \in (0,1), \\[2ex] \left(\dfrac{\sinh[\sqrt{2}(s-1)]}{\sinh\sqrt{2}} + 1-s, -1\right), & s \in (1,2). \end{cases}$$

The maximum condition is as in example 6.2.

Example 6.7. Consider the same problem as example 6.6, with $T = 3$, $|u(s)| \leq 1$ for $s \in [0,3]$, and cost function $J(u) = \int_0^2 x(t)dt$. We do not have to augment this problem; the adjoint function is given by

$$Y(s,t)=\begin{cases} 0 & , \ t<s, \\ e^{(t-s)} & , \ t-1<s\le t, \\ e^{(t-s)}-(t-1-s)e^{(t-1-s)} & , \ t-2<s\le t-1, \\ e^{(t-s)}-(t-1-s)e^{(t-1-s)}+\frac{(t-2-s)^2}{2}e^{(t-2-s)} & , \ t-3<s\le t-2. \end{cases}$$

An optimal trajectory and control are

$$z(t) = \begin{cases} 1-e^{t} & , \quad t\in[0,\omega], \\ 2e^{(t-\omega)}-e^{t}-1 & , \quad t\in[\omega,1], \\ 2e^{(t-\omega)}-e^{t}+(t-2)e^{(t-1)}, & , \quad t\in[1,1+\omega], \\ 2e^{(t-\omega)}-e^{t}+(t-2)e^{(t-1)}-2(t-2-\omega)e^{(t-1-\omega)}-2 & , \quad t\in[1+\omega,2], \\ t-1 & , \quad t\in[2,3]. \end{cases}$$

$$u^*(t) = \begin{cases} -1 & , \quad t\in(0,\omega), \\ +1 & , \quad t\in(\omega,2), \\ 2-t+(t-3)e^{(t-2)}+2e^{(t-1-\omega)}-e^{(t-1)} & , \quad t\in(2,2+\omega) \\ -2(t-3-\omega)e^{(t-2-\omega)}-t+(t-3)e^{(t-2)}+2e^{(t-1-\omega)}-e^{(t-1)} & , \quad t\in(2+\omega,3), \end{cases}$$

where ω is the solution of $(e^{1}+\omega)e^{(1-\omega)} = \frac{1}{2}[3+e^{2}]$, and is given approximately by $\omega = 0.531$, $e^{(1-\omega)} = 1.599$.

$$\text{Let } \alpha^{o} = -1, \quad \lambda(\theta) = \begin{cases} -\alpha^{1} & , \quad \theta\in(-\infty,-1], \\ 0 & , \quad \theta\in(-1,\infty); \end{cases}$$

$$\text{then} \quad \psi(s) = \begin{cases} [\alpha^1-1][e^{(2-s)}-(1-s)e^{(1-s)}]-e^{(1-s)}+2, & s \in (0,1), \\ [\alpha^1-1]e^{(2-s)}+1 & , \; s \in (1,2), \\ 0 & , \; s \in (2,3). \end{cases}$$

α^1 is given approximately by $\alpha^1 = 0.889$. The maximum condition is $\psi(s)v \leq \psi(s)u^*(s)$ a.e. on $(0,3)$, for all $v \in [-1,1]$.

Remark 6.4. Example 6.2 was first solved for the equivalent problem of driving to the point $x(2) = 2$ with the cost function

$$\begin{aligned} J(u) &= \int_0^2 u^2(t)dt + \int_1^2 [1+\dot{x}(t)]^2 dt \\ &= 2\int_0^2 \{u^2(t)+u(t)\}dt + 2\int_1^2 u(t-1)u(t)dt+1 \\ &= \int_0^1 \dot{x}^2(t)dt + \int_1^2 \{[\dot{x}(t)-\dot{x}(t-1)]^2 + [1+\dot{x}(t)]^2\}dt, \end{aligned}$$

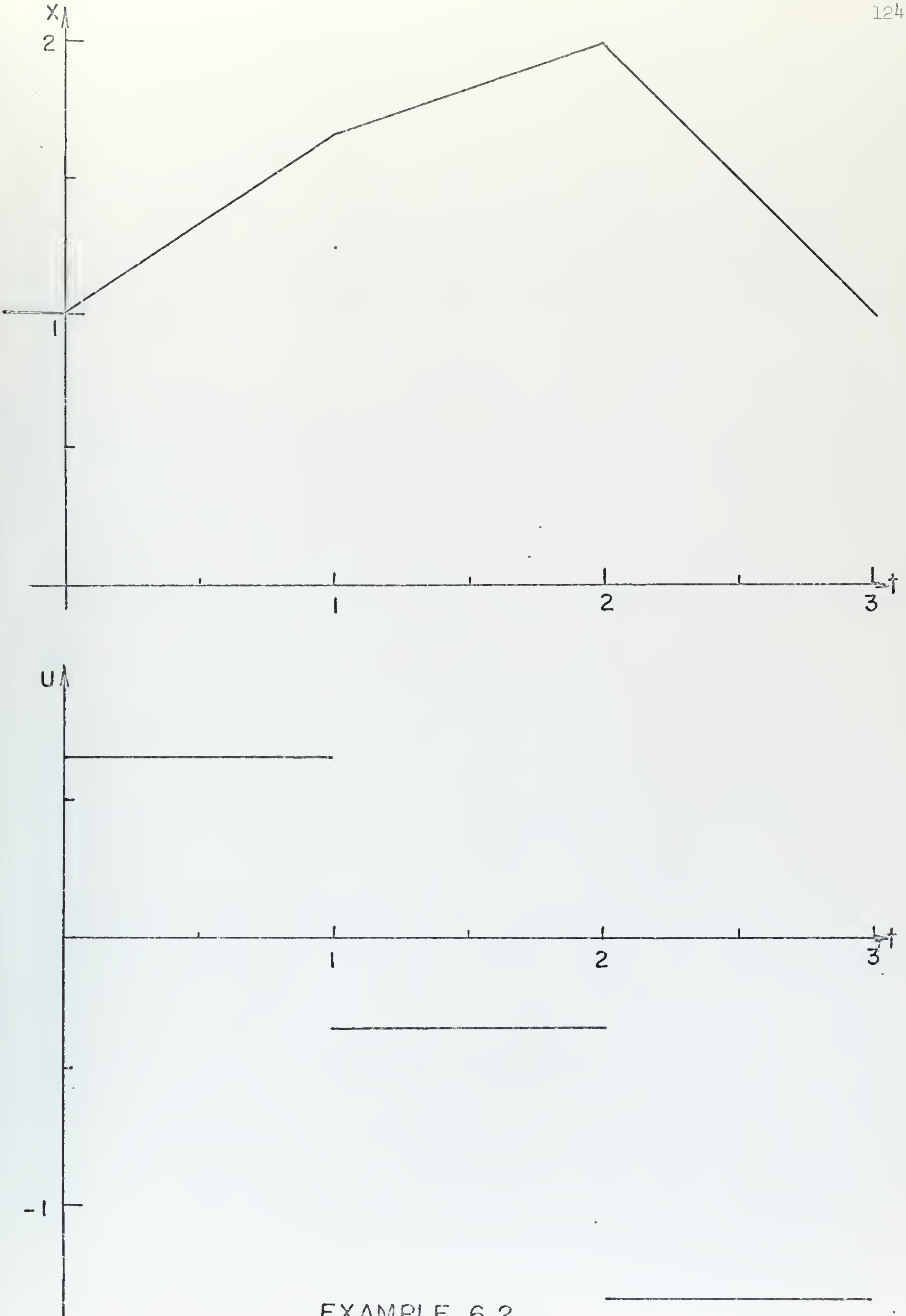
using theorems 4.2 and 5.1 and the information from the Euler equation (see El'sgol'c [9, p. 220]) that extremals consist of straight line segments. This equivalent problem also satisfies the sufficient condition of Hughes [16, section 3]. Knowing the solution to example 6.2, example 6.3 was worked directly with the action of the restraint and theorems 4.3 and 5.2. Examples 6.4 and 6.5 were solved by theorems 4.3 and 5.2 with the assumption that the optimal trajectory would consist of straight line segments. Example 6.6 was solved in the equivalent form of driving to the point $x(1) = 1$ with the cost function

$$\begin{aligned} J(u) &= \int_0^1 \{u^2(t) + [x(t)-t]^2\}dt \\ &= \int_0^1 \{[\dot{x}(t)-x(t)]^2 + [x(t)-t]^2\}dt, \end{aligned}$$

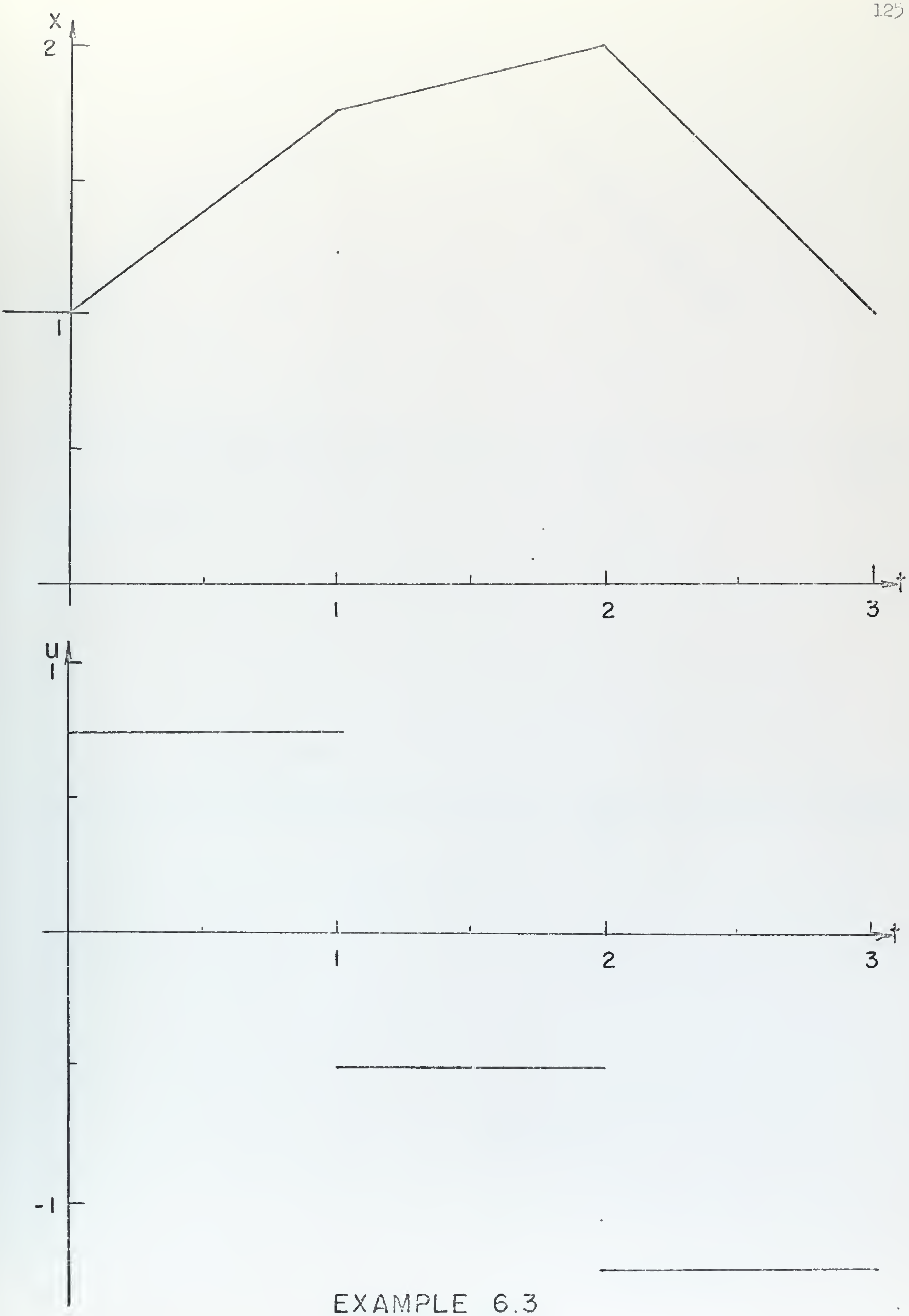
using the Euler equations and theorems 4.2 and 5.1. The solution in example 6.7 was determined by inspection and checked with theorems 4.2 and 5.1. Examples 6.2, 6.6, and 6.7 were solved before the usable form of theorem 4.3 was determined; unsuccessful attempts were made to show that these examples satisfied theorem 4.4 with $\alpha^0 = -1$. One probably will not be able to solve problems using theorems 4.3 and 5.1 alone; equivalent problems in the form of problem 4.2 (possibly requiring a bounded state variables constraint) or the calculus of variations can provide a great deal of information.

Remark 6.5. Pontryagin, Boltyanskii, Gamkrelidze, and Mishchenko [23, chapter VI] point out that in problems with bounded state variables the multiplier function generally has jumps. These occur when the optimum trajectory enters or leaves the boundary and when the optimum trajectory follows the boundary across a point where the boundary is not smooth. Thus we expect the jumps due to the endpoints of the target functions in the examples above, and the jump due to the corner in the middle of the target function in example 6.5. Since the multiplier satisfies an equation similar to the equation for $Y(s,t)$, we expect the jumps to be propagated once they appear. The extra jump in example 6.4 comes from the trajectory and the relationship between T and h. One normally expects $\dot{z}$ to be discontinuous at t_0; if the equation is of neutral type this discontinuity is propagated forward in time. If T is not an integral multiple of h, the discontinuity will fall in the interior of $[T-h,T]$; if ζ is smooth at that point, the control must be discontinuous to

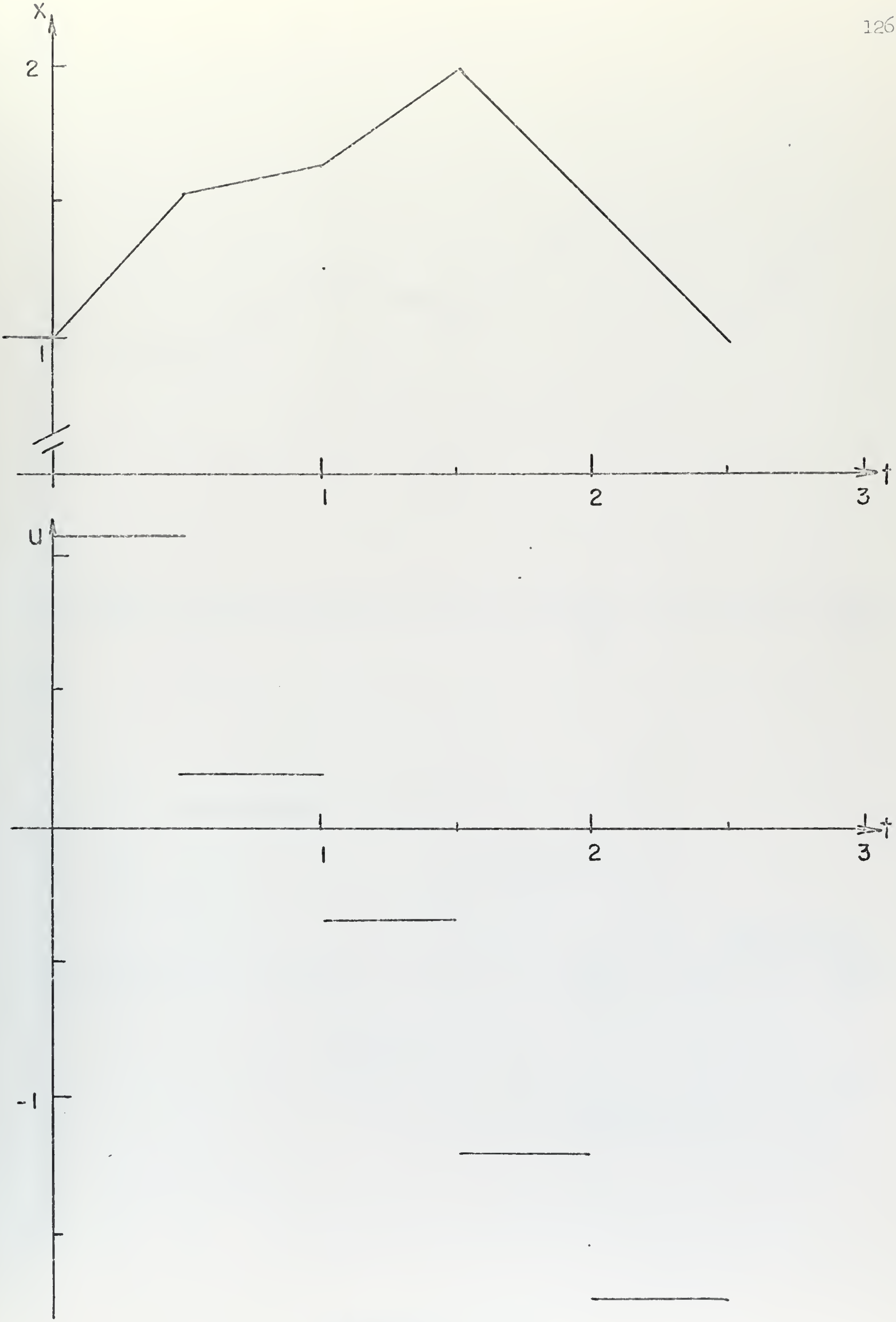
block the propagation of the discontinuity in $\dot{z}$. Since ψ determines u, this requires a corresponding discontinuity in ψ.



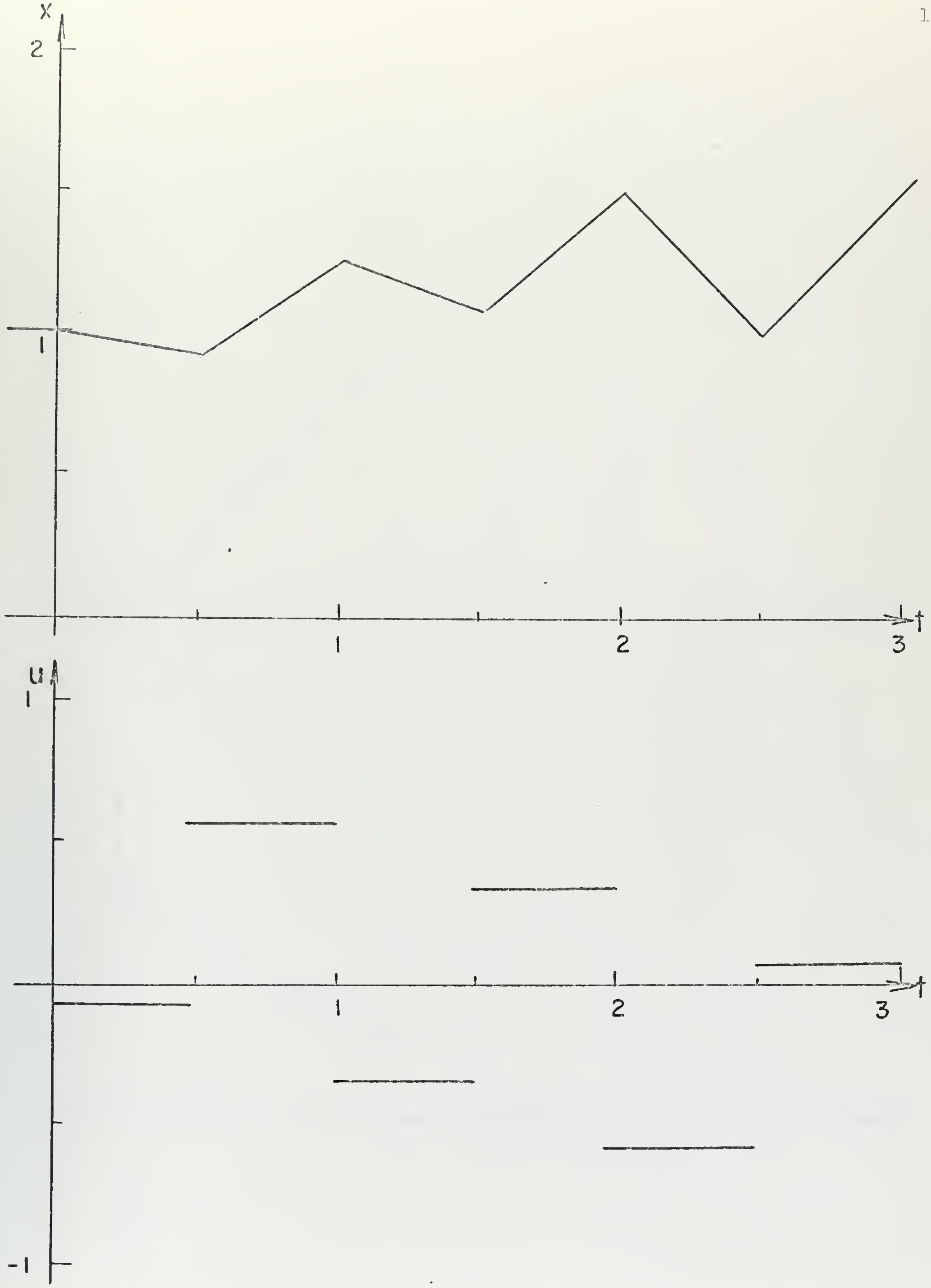
EXAMPLE 6.2



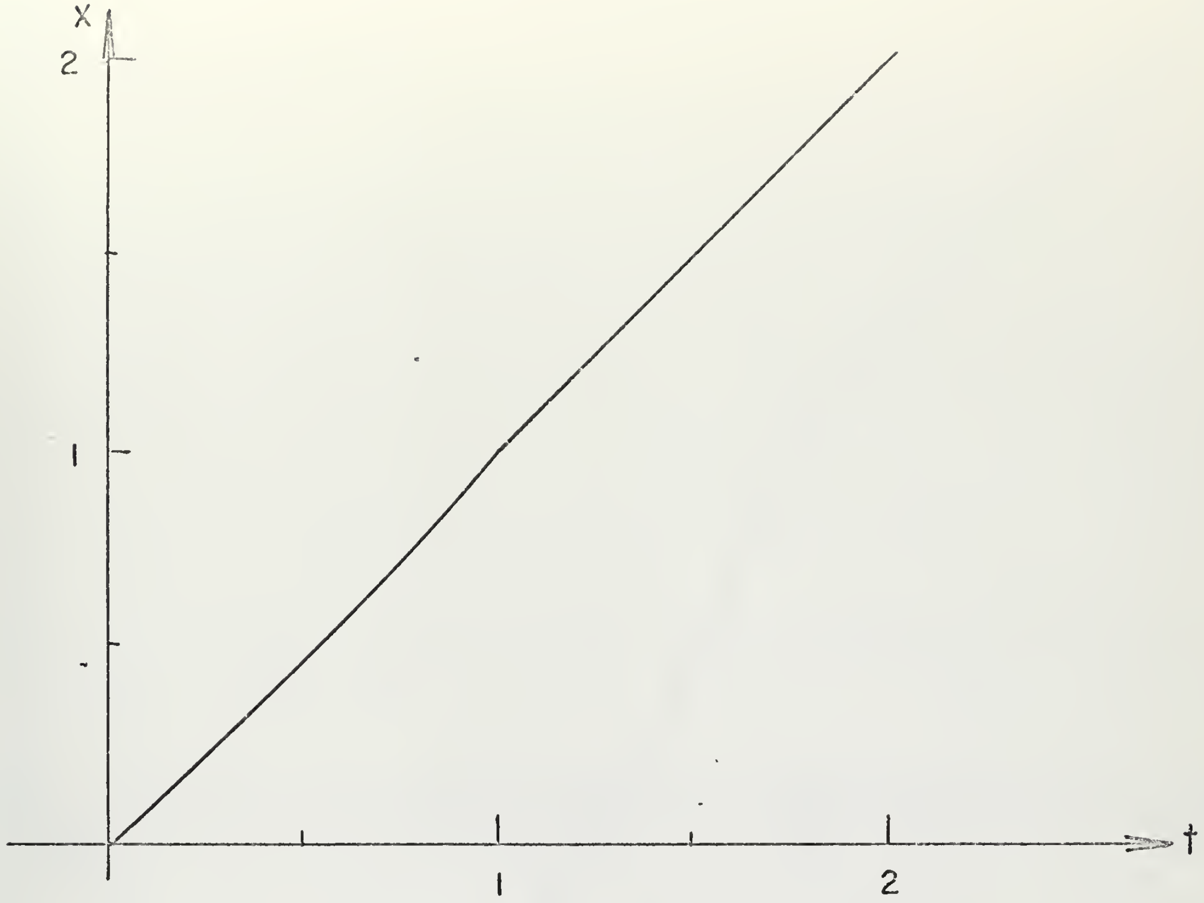
EXAMPLE 6.3

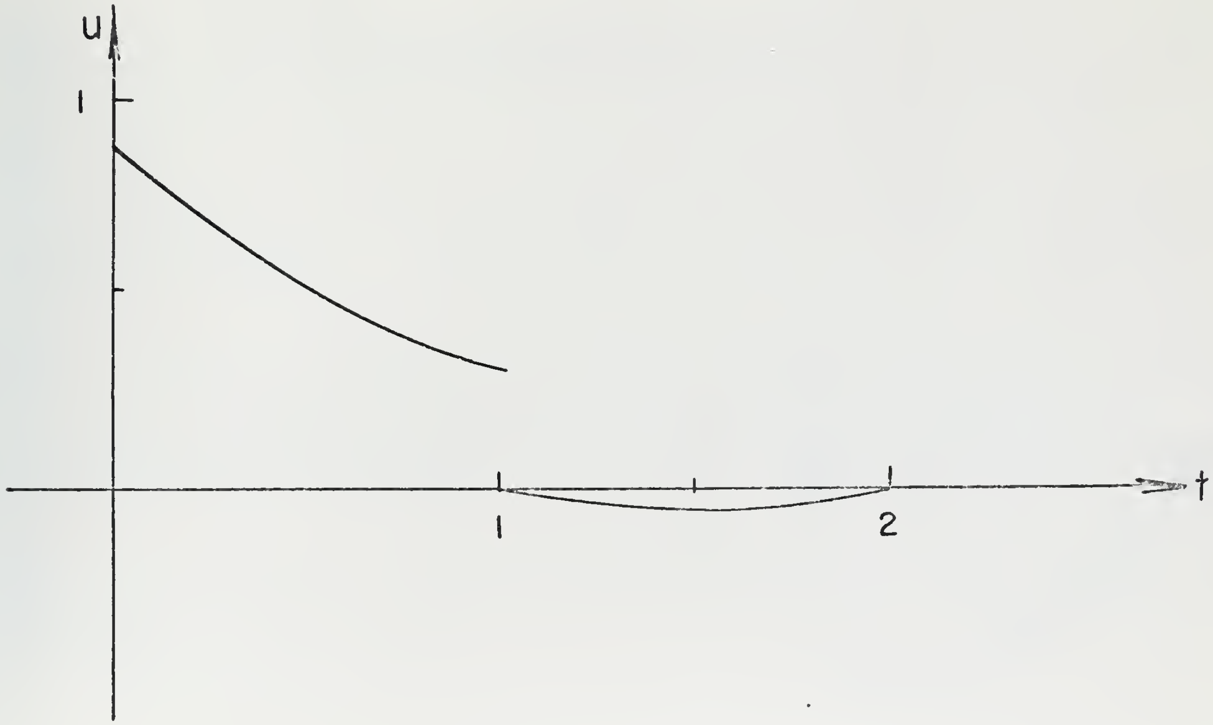


EXAMPLE 6.4

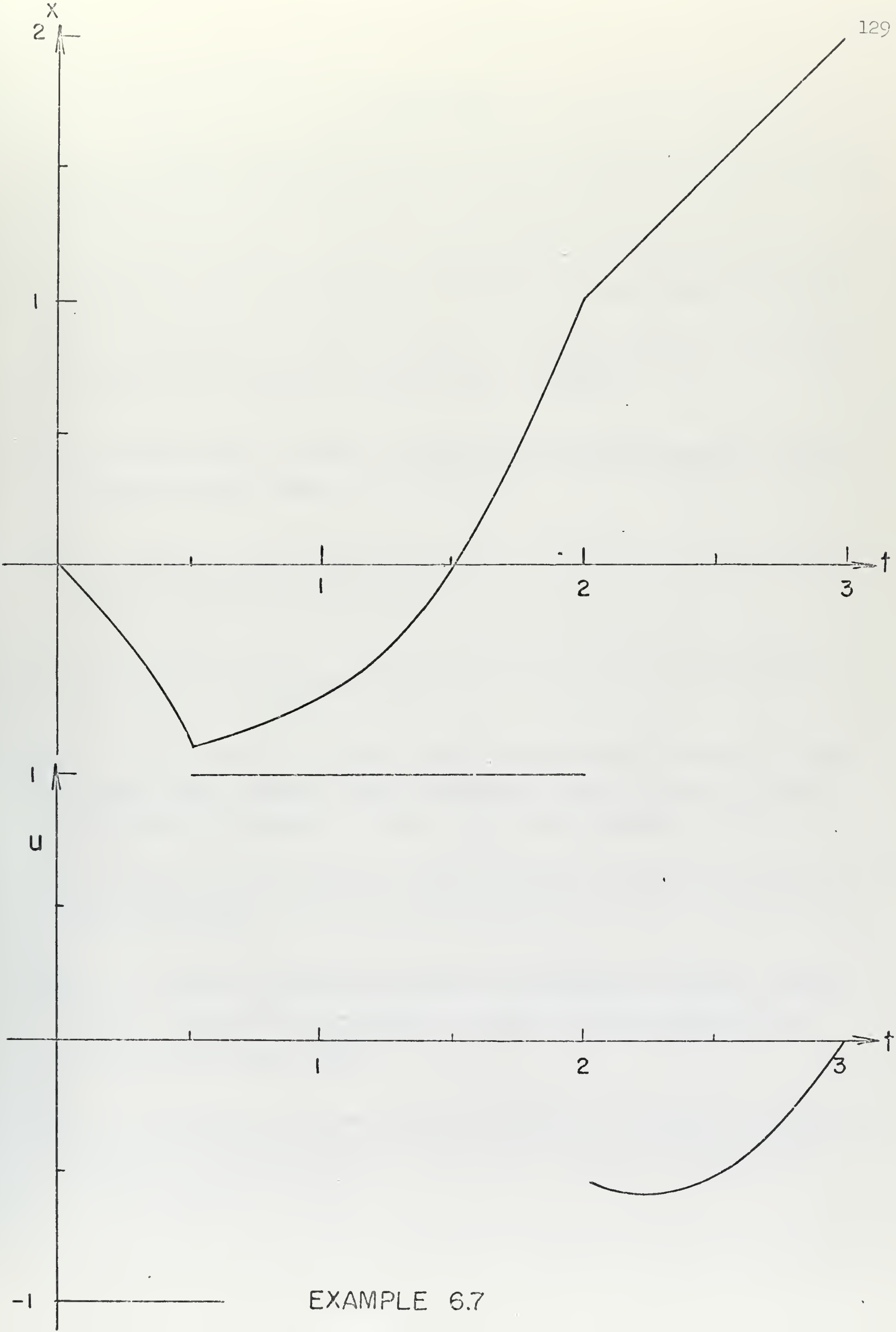


EXAMPLE 6.5





EXAMPLE 6.6



EXAMPLE 6.7

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